Subscription, $1.50 Per Year in Advance *Single Copies, 20c*

VOL. XIII UNIVERSITY, LA., APRIL, 1939 No. 7

Entered as second-class matter at University, Louisiana.

Published monthly excepting June, July, August, September, by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

☞All Business Communications should be addressed to the Editor and Manager, University, La., or P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.

2. To supply an additional medium for the publication of expository mathematics.

3. To promote more scientific methods of teaching mathematics.

4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

On the Applications of Mathematics

As mathematical research becomes more highly specialized we must expect the research mathematician more and more to leave to others the task of finding the applications of his research—if haply any should exist. To one who may regard the applications of mathematics as of paramount importance to the mere discovery of mathematical truth, this relegation, to those inexpert in the science, of the task of discovering its practical uses appears questionable. On the other hand, much valuable research might be lost if too much of the mathematician's attention should be diverted to finding uses other than mathematical ones for the mathematics he is busy creating.

But when one considers the vast amount of pure mathematics already created and stored in the libraries of the world, the question naturally arises: Could a commission be created composed of mathematicians, engineers, chemists, physicists, industrialists, and specialists in other fundamental fields, whose associated studies of some of this stored mathematical material might bring to light hitherto undiscovered uses of it? We believe that trends in this direction already exist. Until such trends shall definitely culminate in the organization of such a commission or until some kind of organized cooperation between mathematical technicians and non-mathematical ones shall be brought into existence, the search for applications of mathematical techniques in non-mathematical fields must remain, as in the past, more or less unordered or subject to mere accidental discovery.

S. T. Sanders.

Continued Fractions*

By W. T. Scott and H. S. Wall
Northwestern University

1. Introduction. There is a wealth of material on *continued fractions* and, in writing a brief expository article on this subject, it is impossible to do more than sketch a representative part of the known continued fraction theory.

A discussion of the simple continued fraction is given because of its importance in analytical number theory and in Diophantine approximation. The simplicity and completeness of the theory of simple continued fractions make it well suited for lecture material in graduate and advanced undergraduate courses. Only a few results on this topic can be given here, but adequate references are provided for those who wish to make a more thorough study of the field.

Although there are many types of function-theoretic continued fractions, that is, continued fractions in which the elements are functions of one or more variables, the so-called *corresponding type*† is the one which has been most widely studied. Most of the results which have been obtained for the *corresponding type* are rather heavily restricted, and the general theory is comparatively undeveloped. The researcher will realize that this is a field of tremendous scope and almost unlimited opportunity. It was in this field that Stieltjes wrote one of the most celebrated papers in mathematics, his memoir on the moment problem.‡

In this one paper Stieltjes defined a new and useful type of integral, proved an important theorem on analytic functions, and obtained a complete solution of the moment problem which bears his name.

2. Fundamental Formulas and Definitions. A terminating continued fraction is an expression of the form

$$b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+\cdots\,\cfrac{a_{n-1}}{b_{n-1}+\cfrac{a_n}{b_n}}}}}$$

*This is the first of a series of expository articles written at the invitation of the Editors.

†Perron: *Die Lehre von den Kettenbrüchen,* (Second Edition), p. 301ff.

‡This paper was published first in the Annales de Toulouse, Vol. 8 (1894), then in Mémoires présentés par divers savants a l'Académie National de France, Vol. 32, and finally in Stieltjes' Oeuvres, Vol. 2.

where the elements a_k and b_k are real or complex numbers and $a_k \neq 0$, $k=1,2,3,\cdots,n$. To conserve space we shall write the continued fraction in the more compact form

$$(1.1) \qquad b_0+a_1/b_1+a_2/b_2+a_3/b_3+\cdots+a_n/b_n.$$

It is clear that a terminating continued fraction is a rational function of its elements. That is,

$$b_0+a_1/b_1+a_2/b_2+a_3/b_3+\cdots+a_k/b_k=A_k/B_k$$

where A_k and B_k are polynomials in the elements a_i, b_i. In order to determine these polynomials, we calculate the first few directly and find that it is convenient to take

$$(1.2) \qquad A_0=b_0, B_0=1, A_1=b_0b_1+a_1, B_1=b_1.$$

Of course the numerator and denominator of A_k/B_k are determined only to a non-zero common factor. One can then easily verify by mathematical induction that the following fundamental recursion formulas hold, namely,

$$(1.3) \qquad \begin{aligned} A_k&=b_kA_{k-1}+a_kA_{k-2}, \\ B_k&=b_kB_{k-1}+a_kB_{k-2}, \end{aligned}$$

$(k=2,3,4,\cdots,n)$. These formulas constitute the starting point in any investigation of continued fractions. An important and immediate consequence is the identity

$$(1.4) \qquad A_kB_{k-1}-A_{k-1}B_k=(-1)^{k-1}a_1a_2a_3\cdots a_k.$$

The fraction A_k/B_k is called the kth *convergent* of the continued fraction, and if $B_n \neq 0$, the value of (1.1) is A_n/B_n.

A non-terminating continued fraction is an expression of the form

$$(1.5) \qquad b_0+a_1/b_1+a_2/b_2+a_3/b_3+\cdots+a_n/b_n+\cdots$$

where $a_k \neq 0$, $k=1,2,3,\cdots$. In order to assign a meaning to this symbol, we turn to the sequence of convergents:

$$(1.6) \qquad A_0/B_0,\ A_1/B_1,\ A_2/B_2,\ A_3/B_3,\cdots.$$

If this sequence has a finite limit L, we say that (1.5) is *convergent* and that its *value* is L. In determining this limit we agree to neglect any *finite* set of convergents A_n/B_n for which $B_n=0$, (note that by (1.4) $A_n \neq 0$ when $B_n=0$). If the sequence (1.6) fails to converge to a finite limit, the continued fraction is said to *diverge*. This occurs in particular if $B_n=0$ for infinitely many values of n. If $\lim_{n\to\infty} (A_n/B_n)$ exists, finite

or infinite, it is convenient to say that (1.5) "converges at least in the wider sense."

One of the simplest sufficient conditions for the convergence of (1.5) is that*

$$|b_n| \geqq |a_n| + 1, \quad n = 1,2,3, \cdots.$$

From this condition, other more convenient tests can be derived. For example the condition

$$|a_n/b_nb_{n-1}| \leqq \tfrac{1}{4}, \quad n = 2,3,4, \cdots, \tag{1.7}$$

is sufficient to insure the convergence of (1.5). Another sufficient condition due to Leighton† is analogous to the "ratio-test" for infinite series, and states that (1.5) converges at least in the wider sense if

$$|a_{n+1}b_{n-1}/a_nb_{n+1}| \leq r < 1$$

from and after some value of n; and diverges (by oscillation) if this ratio is $\geq 1/r > 1$ from and after some value of n; here r is a constant independent of n.

A sufficient condition for divergence, by oscillation, in the case $a_n = 1$, $n = 1,2,3, \cdots$, is the convergence of the series $\sum |b_n|$.‡

Numerous papers on the important question of the convergence of continued fractions have been written, of which we mention the following: (1) Van Vleck, *On the convergence of continued fractions with complex elements,* T. A. M. S. Vol. 2 (1901);(2) Szàsz, *Ueber die Erhaltung der Konvergenz unendlicher Kettenbrüche bei independenter Veränderlichkeit aller inrer Elemente,* J. f. Math. Vol. 147 (1917); (3) Leighton and Wall, *On the transformation and convergence of continued fractions,* American Journal of Mathematics, Vol. LVIII (1936).

3. *Simple Continued Fractions.* A *simple* continued fraction arises naturally from the following algorithm for approximating to an arbitrary real number w_0. Let us assume for the moment that w_0 is not a rational number, and choose the largest integer a_0 which does not exceed w_0, so that $0 < w_0 - a_0 < 1$, or $w_0 = a_0 + 1/w_1$, where $w_1 > 1$. Choose next the largest integer a_1 which does not exceed w_1, so that $0 < w_1 - a_1 < 1$ or $w_1 = a_1 + 1/w_2$, where $w_2 > 1$. Continuing in this manner we determine step by step a sequence $w_1, w_2, w_3, \cdots$ of real numbers >1 and integers $a_0, a_1, a_2, \cdots$, all $\geqq 1$ except possibly a_0, such that

$$w_k = a_k + 1/w_{k+1}, \quad k = 0,1,2, \cdots. \tag{2.1}$$

*Perron: *Loc. cit. Die Lehre von den Kettenbrüchen,* (Second Edition), p. 254.
†Walter Leighton: B. A. M. S., Vol. 42 (1936). Abstract No. 340
‡Perron, *loc cit.*, p. 235.

This process cannot terminate. For if at some stage w_k is an integer a_k, (necessarily $\geqq 2$), we would then have, by successive substitution:

$$w_0 = a_0 + 1/a_1 + 1/a_2 + \cdots + 1/a_k, \tag{2.2}$$

which is a rational number. Hence if w_0 is not rational we obtain the *formal equality*

$$w_0 = a_0 + 1/a_1 + 1/a_2 + \cdots + 1/a_k + \cdots . \tag{2.3}$$

In the continued fraction so obtained, a_0 is an integer which may be positive, negative, or 0, and $a_1, a_2, a_3, \cdots$ are all positive integers. Such a continued fraction is called an *infinite simple continued fraction.* The continued fraction in (2,2) is called a *finite or terminating simple continued fraction.*

We have spoken of (2.3) as a formal equality. It is easy to show that it is a true equality (in a sense of limits), that is

$$\lim_{n\to\infty} (A_n/B_n) = w_0, \tag{2.4}$$

where A_n/B_n is the nth convergent. To do this, note that if we carry out the successive substitution process in (2.1) only to a finite number of steps, we obtain an equality $w = a_0 + 1/a_1 + 1/a_2 + \cdots + 1/w_k$, which may be written in the form $w_0 = (w_k A_{k-1} + A_{k-2})/(w_k B_{k-1} + B_{k-2})$. Consequently

$$w_0 - \frac{A_{k-1}}{B_{k-1}} = \frac{w_k A_{k-1} + A_{k-2}}{w_k B_{k-1} + B_{k-2}} - \frac{A_{k-1}}{B_{k-1}} = \frac{A_{k-2}B_{k-1} - A_{k-1}B_{k-2}}{B_{k-1}(w_k B_{k-1} + B_{k-2})}$$

Along with formula (1.4) this gives the inequality

$$\left| w_0 - \frac{A_{k-1}}{B_{k-1}} \right| \leq \frac{1}{B_{k-1}(w_k B_{k-1} + B_{k-2})}, \quad k = 2,3,4,\cdots . \tag{2.5}$$

Now from the fundamental relation $B_k = a_k B_{k-1} + B_{k-2}$, $k = 2,3,4,\cdots$, it follows that $B_0, B_1, B_2, \cdots$ are positive integers, and that

$$B_0 < B_1 < B_2 < B_3 \cdots .$$

Furthermore, $w_k > a_k$ and therefore $w_k B_{k-1} + B_{k-2} > B_k$. Therefore we have from (2.5) the simpler inequality

$$\left| w_0 - \frac{A_n}{B_n} \right| < \frac{1}{B_n B_{n+1}}, \quad n = 1,2,3,\cdots, \tag{2.6}$$

which shows, since $\lim_{n\to\infty} B_n = \infty$, that (2.4) holds, and hence (2.3) is a true equality.

Conversely, let us start with an infinite simple continued fraction $a_0+1/a_1+1/a_2+1/a_3\cdots$ in which a_0 is an arbitrary integer, and $a_1, a_2, a_3, \cdots$ are arbitrary positive integers. Then we may show that this continued fraction converges, and since the a_n are real, the value of the continued fraction is a real number. For that purpose we use formula (1.4) and write A_n/B_n in the form

$$\frac{A_n}{B_n}=\frac{A_0}{B_0}+\left(\frac{A_1}{B_1}-\frac{A_0}{B_0}\right)+\left(\frac{A_2}{B_2}-\frac{A_1}{B_1}\right)+\cdots+\left(\frac{A_n}{B_n}-\frac{A_{n-1}}{B_{n-1}}\right)$$

$$=\frac{A_0}{B_0}-\frac{1}{B_0B_1}+\frac{1}{B_1B_2}-\cdots+(-1)^{n-1}\frac{1}{B_{n-1}B_n}.$$

This expresses A_n/B_n as the sum of the first $n+1$ terms of an alternating series in which each term is numerically less than the preceding and in which the general term approaches 0. Hence by a well-known theorem $\lim_{n\to\infty}(A_n/B_n)$ exists and is a finite real number w_0. Moreover, since the error made in stopping with the $(n+1)$th term is numerically less than the first term neglected we see that $|w-(A_n/B_n)|<1/B_nB_{n+1}$. We may prove in a few lines that w_0 cannot be a rational number. In fact, if $w_0=p/q$ where p,q are integers then

$$|(p/q)-(A_n/B_n)|=|(pB_n-qA_n)/qB_n|,$$

where $|pB_n-qA_n|$ is an integer. Moreover $|pB_n-qA_n|<|q|/B_{n+1}$. Since q/B_{n+1} tends to 0 as n tends to ∞ it follows that from and after some value of n, $pB_n-qA_n=0$, or $A_n/B_n=p/q=A_{n+1}/B_{n+1}$ for $n>$some index n_0. But this implies that $A_nB_{n+1}-A_{n+1}B_n=0$, which is impossible by (1.4).

We may now complete the discussion by going back to the beginning and supposing that we start with a rational number w_0. If the process by which the relations (2.1) were obtained does not terminate then we have the equality (2.3) which we have just shown to be impossible when w_0 is rational. Hence we have proved the following theorem:

Theorem 2.1. *Every real number may be represented as a simple continued fraction; and conversely every simple continued fraction represents a real number. The continued fraction terminates if and only if the real number is rational.*

The question of *uniqueness of the representation* of real numbers by continued fractions is an important one, and can be settled very quickly. *For irrational numbers the representation is unique; and for rational numbers the representation is likewise unique if we make the*

agreement that the last partial denominator shall be >1. The argument runs as follows in the case of infinite simple continued fractions. Suppose $a_0+1/a_1+1/a_2+1/a_3+\cdots=a_0'+1/a_1'+1/a_2'+1/a_3'+\cdots$ are equal infinite simple continued fractions. Let

$$w=1/a_1+1/a_2+1/a_3+\cdots.$$

Then it is easy to see that $1/(a_1+1)\leq w\leq 1$. Similarly $w'=1/a_1'+1/a_2'+1/a_3'+\cdots$ satisfies the inequality $1/(a_1'+1)\leq w'\leq 1$. Then we have: $a_0+w=a_0'+w'$. If $a_0>a_0'$ then $a_0-a_0'=w'-w$ is a positive integer, which is impossible. Likewise, the inequality $a_0'>a_0$ is impossible. Hence $a_0=a_0'$, and therefore

$$a_1+1/a_2+1/a_3+\cdots=a_1'+1/a_2'+1/a_3'+\cdots,$$

from which we conclude as before that $a_1=a_1'$. The proof may now be completed by mathematical induction.

The simple continued fraction is particularly well adapted to computation inasmuch as the convergents A_n/B_n are related to the value w_0 of the continued fraction by the following inequalities:

$$A_0/B_0<A_2/B_2<A_4\,B_4<\cdots w_0\cdots<A_3/B_5<A_3/B_3<A_1/B_1.$$

Hence the successive convergents are alternately less than and greater than the limiting value w_0. This readily follows from (1.3), (1.4). For example, the simple continued fraction for e, the base of natural logarithms, is*

$$\begin{aligned} e=2+1/1+1/2+1/1+1/1+1/4 \\ +1/1+1/1+1/6+1/1+1/1+1/8+\cdots \end{aligned}$$

in which the law of formation of partial quotients is evident. The 11th convergent is 2721/1001, which differs from e by at most 1/8,552,544. The simple continued fraction for π, *of which the law of formation of the partial quotients is not known*, was given to 33 partial quotients by Wallis† as follows:

$$\begin{aligned} \pi=3+1/7+1/15+1/1+1/292+1/1+1/1+1/1+1/2 \\ +1/1+1/3+1/1+1/14+1/2+1/1+1/1+1/2+1/2+1/2 \\ +1/2+1/1+1/84+1/2+1/1+1/1+1/15+1/3+1/13 \\ +1/1+1/4+1/2+1/6+1/6+1/1+\cdots \end{aligned}$$

*Perron, *loc cit.*, p. 134.
†Perron, *loc cit.*, p. 42.

A method of expanding the *square root* of a positive integer into a simple continued fraction is illustrated by the following example:

$$\sqrt{3}=1+\frac{1}{\left(\dfrac{1}{\sqrt{3}-1}\right)},$$

$$\frac{1}{\sqrt{3}-1}=\frac{\sqrt{3}+1}{2}=1+\frac{1}{\left(\dfrac{2}{\sqrt{3}-1}\right)}$$

$$\frac{2}{\sqrt{3}-1}=\sqrt{3}+1=2+\frac{1}{\left(\dfrac{1}{\sqrt{3}-1}\right)}$$

and hence $\sqrt{3}=1+1/1+1/2+1/1+1/2+1/1+1/2+\cdots$. This is a *periodic* continued fraction.

Let us consider the general periodic simple continued fraction

$$w=a_0+1/a_1+1/a_2+\cdots+1/a_k|+1/b_0+1/b_1+\cdots 1/b_p+1/b_0 \\ +1/b_1+\cdots+1/b_p+1/b_0+\cdots,$$

where the periodicity begins after the kth partial quotient. Let

$$W=b_0+1/b_1+1/b_2+\cdots+1/b_p+1/b_0+1/b_1+\cdots+1/b_p+1/b_0+\cdots,$$

so that

$$W=b_0+1/b_1+1/b_2+\cdots+1/b_p+1/W=(A'_pW+A'_{p-1})/(B'_pw+B'_{p-1}),$$

where A_n'/B_n' is the nth convergent of $b_0+1/b_1+1/b_2+1/b_3+\cdots$. Hence we see that W is a quadratic irrational number, i e., a root of a quadratic equation with integral coefficients. Inasmuch as

$$w=(A_kW+A_{k-1})/(B_kW+B_{k-1})$$

it is easy to see that w is likewise a quadratic irrational number. Hence we have this result:

Theorem 2.2. *Every periodic simple continued fraction represents a quadratic irrational number.*

The *converse* of this theorem is also true, namely: *every real quadratic irrational number has a periodic simple continued fraction representation.**

We shall conclude this discussion by mentioning two other applications of simple continued fractions. One grows out of the inequality

*Perron, *loc. cit.*, p. 74.

(2.6). Since $B_{n+1}>B_n$ we can state that if w is an arbitrary irrational number, then an infinite number of distinct rational fractions p/q can be found such that $|w-(p/q)|<k/q^2$, when $k=1$. This introduces a new problem, namely the problem of finding the smallest value of k for which the above theorem is true. Hurwitz* showed that the smallest value of k is $1/\sqrt{5}$. In doing this, he used simple continued fractions.

The second application of simple continued fractions which we wish to mention is to the diaphantine equation of Pell, namely $p^2-Dq^2=1$ in which D is a positive integer not a perfect square, and p,q are integers to be determined. It can be shown† that the solutions of Pell's equation consist of the integers $p=A_{nk-1}$, $q=B_{nk-1}$, where A_k/B_k is the kth convergent of the simple continued fraction for $\sqrt{D}$, and k is the number of elements of the period of the continued fraction.

In the next section we shall give a detailed account of how simple continued fractions may be used to prove the existence of transcendental numbers.

4. Transcendental Numbers. The ancient Greeks knew of the existence of irrational numbers. For example, they knew that the diagonal of a unit square has a length which is not a rational number; and they suspected that the ratio π of the circumference of a circle to its diameter is not rational. We know now that π is not rational; and we know in addition that π and $\sqrt{2}$ are very different sorts of numbers. Indeed $\sqrt{2}$ is a root of an *irreducible*‡ algebraic equation with rational coefficients, namely $x^2-2=0$, but π is not the root of such an equation. This last statement is not easy toprove. By means of simple continued fractions however we can prove the existence of *transcendental numbers*, i. e., of numbers which, like π, are not roots of algebraic equations with rational coefficients. And, moreover, we have a means for constructing such numbers to any desired number of decimal places.

To do this we first prove the following lemma.

Lemma (*Liouville*). *If x_0 is a root of an irreducible algebraic equation of degree $n>1$ with rational coefficients, then there exists a positive number $c<1$ such that for all integers $p,q>0$ the following inequality holds*:

$$|(p/q)-x_0|>c/q^n. \tag{3.1}$$

Proof. The inequality obviously holds for every positive number $c<1$ when p,q are positive integers such that $|(p/q)-x_0|>1$. Hence

*Mathematische Annalen, Vol. 39 (1891) pp. 279-284.

†Perron, *loc. cit.*, p. 102ff.

‡A polynomial with rational coefficients will be called irreducible if it cannot be factored into the product of two polynomials with rational cofficients.

we need consider only those values of p,q for which $|(p/q)-x_0|\leq 1$. Suppose that $f(x)\equiv c_0x^n+c_1x^{n-1}+\cdots+c_n=0$ (c_k integral, $c_0\neq 0$) is the equation which x_0 satisfies. Since $f(x)$ is irreducible, $f(p/q)\neq 0$, and consequently

$$(3.2)\qquad |f(p/q)|=\frac{|c_0p^n+c_1p^{n-1}q+\cdots+c_nq^n|}{q^n}\geq 1/q^n,$$

since the numerator is a positive integer at least as great as 1. Since $f(x_0)=0$ we have, by the mean value theorem of differential calculus:

$$f(p/q)=f(p/q)-f(x_0)=((p/q)-x_0)f'(\xi)$$

where ξ is a properly chosen number between p/q and x_0. This with (3.2) gives

$$(3.3)\qquad |((p/q)-x_0)f'(\xi)|\geq 1/q^n.$$

Since ξ lies between p/q and x_0 and since $|(p/q)-x_0|\leq 1$ it follows that $|\xi|<|x_0|+1$ and therefore

$$\begin{aligned}|f'(\xi)|&\leq|nc_0\xi^{n-1}|+|(n-1)c_1\xi^{n-2}|+\cdots+|c_{n-1}|\\&<|nc_0(|x_0|+1)^{n-1}|+|(n-1)c_1(|x_0|+1)^{n-2}|+\cdots+|c_{n-1}|.\end{aligned}$$

Hence $|f'(\xi)|$ is less than a fixed number independent of p,q, say $|f'(\xi)|<1/c$, and we may surely take $c<1$. This with (3.3) gives (3.1), as was to be proved.

With the aid of this lemma we may now construct transcendental numbers. For that purpose let $b_0+1/b_1+1/b_2+1/b_3\cdots$ be a simple continued fraction with kth convergent A_k/B_k, and value x_0. Suppose the positive integers b_k have been so chosen that for every positive integer n however large an index k can be found such that

$$(3.4)\qquad b_{k+1}>B_k^n.$$

This can evidently be done in infinitely many ways. One way is to take $b_{k+1}=1+B_k^k$. *Under these conditions x_0 is a transcendental number.* To show this we need but show that however small $c>0$ be chosen and however large n be chosen, positive integers p,q can be found for which (3.1) fails to hold. Since (3.4) holds by hypothesis for at least one value of k and for every value of n, it must hold for infinitely many values of k for each fixed n. Suppose now if possible that x_0 satisfies an irreducible algebraic equation of degree n with rational coefficients. Then by (2.6) and (1.3):

$$\left|\frac{A_k}{B_k}-x_0\right|<\frac{1}{B_kB_{k+1}}<\frac{1}{b_{k+1}B_k^2}$$

so that by (3.4)

$$\left| \frac{A_k}{B_k} - x_0 \right| < \frac{1}{B_k^2 B_k^n}$$

for infinitely many values of k. But no matter how small we choose $c>0$, $1/B_k^2$ will be less than c for all sufficiently large values of k. Hence by the lemma, x_0 must be a transcendental number

This is historically the first proof of the existence of transcendental numbers, and is due to Liouville.* The numbers given by this construction are known as *Liouville numbers.* A remarkable property of Liouville numbers is that if x_0 is a Liouville number, then $(ax_0+b)/(cx_0+d)$ is a Liouville number for arbitrary integral values of a,b,c,d with $ad-bc\neq 0$. It follows that the Liouville numbers are everywhere dense in the set of real numbers.

5. *Operations on Continued Fractions.* There is no simple way in which two continued fractions can be added or multiplied together to form a sum or product continued fraction. Neither can the operations of differentiation or integration be performed upon continued fractions with variable elements. However there are certain operations which can be performed. For example, if $c_1, c_2, c_3, \cdots$ are $\neq 0$,

$$b_0+c_1a_1/c_1b_1+c_1c_2a_2/c_2b_2+c_2c_3a_3/c_3b_3+\cdots \tag{4.1}$$

has convergents which are independent of the values of $c_1, c_2, c_3, \cdots$. We mention also the useful operations of "contraction" and "extension".† A general treatment of the question of transformation of continued fractions was given by Leighton and Wall (*loc. cit*).

In this section we shall describe an interesting *group* of transformations of continued fractions. We shall take a continued fraction of the form

$$\xi = x_0 - x_1/1 - x_2/1 - x_3/1 - \cdots \tag{4.2}$$

where $x_0, x_1, x_2, \cdots$ are real or complex numbers and $x_k \neq 0$ for $k>0$. If the b_i in (4.1) are $\neq 0$ for $i>0$ it is always possible to choose the c_i in such a way as to throw (4.1) into the form (4.2). Let

$$A_0/B_0,\ A_1/B_1,\ A_2/B_2, \cdots$$

be the sequence of convergents of (4.2), and let us consider the problem of constructing a continued fraction

$$\xi' = b_0 + a_1/b_1 + a_2/b_2 + a_3/b_3 + \cdots$$

*J. de Math. Vol. 16, (1851).
†Perron, *loc cit.*, pp. 197-205.

with these same convergents but in the order

$$A_1/B_1,\ A_0/B_0,\ A_3/B_3,\ A_2/B_2, \cdots .$$

Let A_n'/B_n' be the nth convergent of ξ'. Then we require that

$$(4.3)\qquad \begin{matrix} A'_{2n}=A_{2n+1},\ B'_{2n}=B_{2n+1}, \\ A'_{2n+1}=A_{2n},\ B'_{2n+1}=B_{2n}, \end{matrix} \qquad n=0,1,2,\cdots .$$

We find in particular that (4.3) implies that

$$b_0=x_0-x_1,\ b_1=1,\ a_1=x_1,\ a_2=1-x_3,\ b_2=-x_2.$$

From (4.3) and the relations

$$A_n'=b_nA'_{n-1}+a_nA'_{n-2},\ B'_n=b_nB'_{n-1}+a_nB'_{n-2}$$

we have

$$A_{2n+1}=b_{2n}A_{2n-2}+a_{2n}A_{2n-1},$$

$$B_{2n+1}=b_{2n}B_{2n-2}+a_{2n}B_{2n-1},$$

$$A_{2n}=b_{2n+1}A_{2n+1}+a_{2n+1}A_{2n-2},$$

$$B_{2n}=b_{2n+1}B_{2n+1}+a_{2n+1}B_{2n-2}.$$

If $x_{2n+1}\neq 1$, $n=1,2,3,\cdots$, these equations may be solved for the a_k,b_k, and with aid of (1.4) we find that $b_{2n}=-x_{2n}$, $a_{2n}=(1-x_{2n+1})$, $b_{2n+1}=1/(1-x_{2n+1})$, $a_{2n+1}=x_{2n}x_{2n+1}/(1-x_{2n+1})$, $n=1,2,3,\cdots$. Finally, put $x_0'=b_0$, $x_1'=-a_1/b_1$, $x_2'=-a_2/b_1b_2$, $x_3'=-a_3/b_2b_3,\cdots$, and ξ' takes the following form:

$$(4.4)\qquad \xi'=x_0'-x_1'/1-x_2'/1-x_3'/1-\cdots,$$

where

$$(4.5)\qquad \begin{cases} x_0'=x_0-x_1,\ x_1'=-x_1,\ x_2'=(1-x_3)/x_2, \\ x'_{2n}=(1-x_{2n-1})(1-x_{2n+1})/x_{2n}, \quad n=2,3,4,\cdots, \\ x'_{2n+1}=x_{2n+1},\ n=1,2,3,\cdots . \end{cases}$$

The equations (4.5) may be considered as constituting a transformation of ξ into ξ'. This transformation cannot affect the convergence or divergence of the continued fraction, nor its value in case it converges. In particular, if

$$(4.6)\qquad |x_{2n+1}|\leqq \tfrac{1}{4}, \qquad |x_{2n}|\geqq 25/4, \qquad n=1,2,3,\cdots$$

then we see that $|x_n'|\leqq\frac{1}{4}$, $n=2,3,4,\cdots$, so that ξ', and therefore ξ, converges by the condition (1.7). In this way we obtain a convergence criterion for ξ, namely (4.6).*

*Leighton and Wall, *loc. cit.*

The transformation (4.5) may be used to obtain the *value* of a certain continued fraction. Note that we have the formal expansion: $1=(1-x_3)^{1/2}/(1-x_3)^{1/2}=(1-x_3)^{1/2}/[1+((1-x_3)^{1/2}-1)]$ so that

$$(4.7)\qquad 1=(1-x_3)^{1/2}/1-x_3/1+(1-x_3)^{1/2}(1-x_5)^{1/2}/1-x_5/1 \\ +(1-x_5)^{1/2}(1-x_7)^{1/2}/1-\cdots$$

for arbitrary $x_3,x_5,x_7,\cdots$ different from 0,1. If we apply the transformation (4.6) to the continued fraction

$$x_0-x_1/1+(1-x_3)^{1/2}/1-x_3/1-(1-x_3)^{1/2}(1-x_5)^{1/2}/1-x_5/1-\cdots$$

the latter becomes

$$x_0-x_1+x_1/1+(1-x_3)^{1/2}/1-x_3/1-(1-x_3)^{1/2}(1-x_5)^{1/2}/1-x_5/1-\cdots.$$

Let us suppose that the continued fraction (4.7) converges and that its value is $v\neq -1$. Then we see that $x_0-x_1/(1+v)=x_0-x_1+x_1/(1+v)$ so that $v=1$. *Hence we conclude that if* (4.7) *converges its value must be* 1 *or* -1.

The transformation which we have described is the simplest one of a *group* of transformations which can be obtained by a similar procedure. For details see the papers on this subject by Wall in B. A. M. S., Vol. 40 (1934), pp. 587-592 and Vol. 44 (1938), pp. 94-99. We shall merely record here convenient formulas which can be used to obtain the transformations.

Let $B_{s,t}$ be the sth convergent of the continued fraction

$$1-x_{1+t}/1-x_{2+t}/1-x_{3+t}/1-\cdots.$$

Let a, b, c, d, be distinct positive integers, put

$$f(p,q)=B_{|c-a|-1,nm+p}/B_{d-b-1,mn+b}/B_{|c-b|-1,mn+q}B_{d-a-1,nm+a}$$

and determine r_{abcd} by the following conditions:

(1) $r_{abcd}=r_{badc}=r_{cdab}=r_{dcba}$;

(2) $r_{abcd}=f(a,b)$ if $a<b<c<d$ or $b<a<c<d$;

(3) $r_{abcd}=-x_{nm+c+1}x_{nm+c+2}\cdots x_{nm+b}f(a,c)$ if $a<c<b<d$;

(4) $r_{abcd}=-f(c,b)/x_{nm+c+1}x_{nm+c+2}\cdots x_{nm+a}$ if $b<c<a<d$;

(5) $r_{abcd}=f(c,c)/x_{nm+b+1}x_{nm+b+2}\cdots x_{nm+a}$ if $c<b<a<d$;

(6) $r_{abcd}=x_{nm+a+1}x_{nm+a+2}\cdots x_{nm+b}f(c,c)$ if $c<a<b<d$.

In these formulas n is a fixed integer $\geqq 3$ and $m=0,1,2,\cdots$.

Now let $i_1, i_2, i_3,\cdots,i_n$ be any permutation of $1,2,3,\cdots,n$ and define $i_{n+s}=n+i_s$, $s\geqq 1$. Then the transformation of ξ into a continued fraction ξ' with the same convergents as ξ but in the order

$$A_{i_1-1}/B_{i_1-1},\ A_{i_2-1}/B_{i_2-1},\ A_{i_3-1}/B_{i_3-1},\cdots$$

is given by the formulas:

$$(4.8)\quad \begin{cases} x'_{nm+s-1} = r_{i_{s-1}i_{s-2}i_s i_{s-3}}, \; s=4,5,\cdots,2n, m=0,1,2,\cdots; \\ x_0' = A_{i_1-1}/B_{i_1-1}, \\ x_1' = (A_{i_1-1}/B_{i_1-1}) - (A_{i_2-1}/B_{i_2-1}), \\ x_2' = [(A_{i_3-1}/B_{i_3-1}) - (A_{i_2-1}/B_{i_2-1})] \\ \qquad\qquad /[(A_{i_3-1}/B_{i_3-1}) - (A_{i_1-1}/B_{i_1-1})]. \end{cases}$$

As an illustration, let $n=3$, $i_1=3$, $i_2=2$, $i_3=1$. For abbreviation put $x=x_{3m+3}$, $y=x_{3m+4}$, $z=x_{3m+5}$. Then

$$x' = \frac{1-x-y-z+xz}{1-y-z}, \quad y' = \frac{xyz}{(1-x-y)(1-y-z)},$$

$$z' = \frac{1-x-y-z+xz}{1-x-y},$$

$m=0,1,2,\cdots$, with certain irregularities in x_0', x_1', x_2'.

For each value of n there are $n!$ such transformations. These form a group of Cremona transformations equivalent to the cross-ratio group of E. H. Moore.

6. Function-Theoretic Continued Fractions. Frobenius* investigated the problem of approximating to a formalpower series $P(x) = c_0 + c_1x + c_2x^2 + \cdots$ $(c_0 \neq 0)$ in the neighborhood of $x=0$ by means of a rational fraction $N_{m,n}(x)/D_{m,n}(x)$ in which the degrees of numerator and denominator do not exceed n and m, respectively $(n,m=0,1,2,3,\cdots)$. He found that for a given n and m there is one and only one such rational fraction in which $D_{n,m}^{(x)} \not\equiv 0$ such that the formal power series $P(x)\ D_{n,m}(x) - N_{n,m}(x)$ begins with the $(m+n+1)$th or a higher power of x. Pade'† arranged these fractions in the following table of double entry known as the Pade' table:

$$\begin{array}{llll} \dfrac{N_{0,0}}{D_{0,0}}, & \dfrac{N_{0,1}}{D_{0,1}}, & \dfrac{N_{0,2}}{D_{0,2}} & \dfrac{N_{0,3}}{D_{0,3}}, \cdots \\[2ex] \dfrac{N_{1,0}}{D_{1,0}}, & \dfrac{N_{1,1}}{D_{1,1}}, & \dfrac{N_{1,2}}{D_{1,2}}, & \dfrac{N_{1,3}}{D_{1,3}}, \cdots \\[2ex] \dfrac{N_{2,0}}{D_{2,0}}, & \dfrac{N_{2,1}}{D_{2,1}}, & \dfrac{N_{2,2}}{D_{2,2}}, & \dfrac{N_{2,3}}{D_{2,3}}, \cdots \\ \cdot & \cdot & \cdot & \end{array}$$

* *Jour. für Math.* Vol. 90 (1881).
†Pade': Ann. Ec. Norm. Vol. 9, series (3) (1892).

and considered the question of convergence of infinite sequences of approximants chosen from this table. He proved in particular that every infinite sequence of distinct approximants for the function e^x converges for every x to the limit e^x. Montessus* Van Vleck†, Wilson,‡ Wall,§ and others, have studied the corresponding problem for other power series. A remarkable fact in this connection is that *different sequences may have different limits.*

The rôle played by continued fractions in the study of the Pade′ table is quite an essential one. For the horizontal, vertical, diagonal, and stair-like files of approximants in the table constitute the sequences of convergents of four important types of function-theoretic continued fractions. For details, see Perron, *loc cit.*, pp. 477ff.

In a so-called normal Pade′ table the approximants

$$\frac{N_{0,0}}{D_{0,0}},\quad \frac{N_{0,1}}{D_{0,1}},\quad \frac{N_{1,1}}{D_{1,1}},\quad \frac{N_{1,2}}{D_{1,2}},\quad \frac{N_{2,2}}{D_{2,2}},\quad \frac{N_{2,3}}{D_{2,3}},\quad \cdots$$

are the convergents of a continued fraction of the form

$$c_0+a_1x/1+a_2x/1+a_3x/1+\cdots \tag{5.1}$$

where $a_1,a_2,a_3,\cdots$ are constants different from 0. Quite independently of the Pade′ table, we may obtain the continued fraction (5.1) directly from the power series simply by means of repeated long division. Even in case the Pade′ table is not "normal", or, more generally, when the power series is not "semi-normal", this same process leads in every case to a continued fraction of the form

$$c_0+a_1x^{\alpha_1}/1+a_2x^{\alpha_2}/1+a_3x^{\alpha_3}/1+\cdots \tag{5.2}$$

where $a_1,a_2,a_3,\cdots$ are constants $\neq 0$, and $\alpha_1,\alpha_2,\alpha_3,\cdots$ are positive integers. Leighton,‡ Leighton and Scott,|| and Scott and Wall,¶ have studied this general "corresponding type" continued fraction. An example will serve to illustrate how the continued fraction may be obtained.

Let $\quad P(x)=1+x+x^2+x^3+x^5+x^7+x^{11}+x^{13}+\cdots$

where the exponents are the prime numbers in order. Write

$$P(x)=1+x(1+x+x^2+x^4+\cdots)$$

*Rend. di Palermo, Vol. 19 (1905).
†Annals of Math. (2) Vol. 3 (1901); T. A. M. S. Vol. 5 (1904).
‡Proceedings of the London Math. Soc., Ser. (2) Vol. 30 (1929).
§T. A. M. S. Vol. 34 (1932); B. A. M. S. Vol. 38 (1932).
‡B. A. M. S., Vol. 42 (1936), Abstract No. 113.
||B. A. M. S., Vol. 44 (1938), Abstract No. 351.
¶B. A. M. S. Vol. 45 (1939) Abstract No. 59.

and determine the reciprocal of the series in parentheses by ordinary long division. This gives

$$P(x)=1+x/1-x(1-x^2+2x^3-2x^4+x^5+x^6+\cdots).$$

And, after another long division,

$$P(x)=1+x/[1-x/(1+x^2(1-2x+3x^2-5x^2+\cdots))].$$

This laborious process may be used to obtain the continued fraction to any specified number of partial quotients. We find that the first 12 partial quotients are as follows:

$$\begin{aligned}&1+x/1-x/1+x^2/1+2x/1-.5x/1-.5x/1\\&\qquad+2x^2/1-x^3/1+x/1-x/1-2x^5/1-1.5x^2/1+\cdots.\end{aligned}$$

A systematic statement of the process used to get the corresponding continued fraction of an arbitrary power series was given by Leighton and Scott (*loc. cit*). There is no loss in generality if it is assumed that the arbitrary power series is of the form

$$P_0(x)=1+\sum_{i=1}^{\infty} c_i x^i.$$

A set of power series $P_k(x)$ is then defined by power series identities of the form

$$P_{n+1}(x)=a_{n+1}x^{\alpha_{n+1}}/[P_n(x)-1], \quad n=0,1,2,\cdots, \tag{5.3}$$

where a_{n+1} is a complex constant and α_{n+1} is a positive integer, so chosen that $P_{n+1}(0)=1$ provided $P_n(x)\neq 1$. If $P_n(x)\equiv 1$,

$$P_0(x)=1+a_1x^{\alpha_1}/1+a_2x^{\alpha_2}/1+\cdots+a_nx^{\alpha_n}/1, \tag{5.4}$$

but if $P_n(x)\equiv 1$ for no values of n, the process described above will not terminate and we have (with $\sim$ meaning *corresponds* to)

$$P_0(x)\sim 1+a_1x^{\alpha_1}/1+a_2x^{\alpha_2}/1+a_3x^{\alpha_3}/1+\cdots, \tag{5.5}$$

The reader will note that actual equality is used in (5.4) since the continued fraction terminates and hence has a value except possibly for certain isolated values of x. However, (5.5) implies only a correspondence between the power series and the continued fraction. It is known* that, if the continued fraction converges uniformly to an analytic function in the neighborhood of the origin, $P_0(x)$ is the element of this analytic function at the origin, and equality may be used in (5.5) for x near 0.

*Leighton and Scott, *loc cit.*

The reader has already seen that the direct method of expanding a power series into a corresponding continued fraction is laborious. In special cases, notably when the integers α_k in (5.5) are all 1, formulas for the coefficients a_k in the continued fraction may be obtained in terms of the coefficients of the power series.*

In the case of a certain quotient of hypergeometric series the relations (5.3) may be obtained without actually performing the divisions indicated, and hence the corresponding continued fractions for a number of elementary functions may be derived. The starting point is the pair of "contiguous relations":

$$\begin{aligned} F(\alpha,\beta,\gamma;x) &= F(\alpha,\beta+1,\ \gamma+1;x) - \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}\, xF(\alpha+1\beta,+1,\gamma+2;x), \\ F(\alpha,\beta,\gamma;x) &= F(\alpha+1,\beta,\gamma+1;x) - \frac{\beta(\gamma-\alpha)}{\gamma(\gamma+1)}\, xF(\alpha+1,\beta+1,\gamma+2;x), \end{aligned} \tag{5.6}$$

which are easily seen to be satisfied by the hypergeometric series

$$F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \cdots .$$

We then put

$$\begin{aligned} P_{2n+1}(x) &= \frac{F(\alpha+n,\beta+n+1,\gamma+2n+1;x)}{F(\alpha+n+1,\beta+n+1,\gamma+2n+2;x)}, \\ P_{2n}(x) &= \frac{F(\alpha+n,\beta+n,\gamma+2n;x)}{F(\alpha+n)\beta+n+1,\gamma+2n+1,x)}, \end{aligned} \qquad n=0,1,2,\cdots,$$

and obtain at once from (5.6) the relations

$$\begin{aligned} P_{2n+1}(x) &= a_{2n+2}x/[P_{2n}(x)-1], \\ P_{2n+2}(x) &= a_{2n+1}(x)/[P_{2n+1}(x)-1], \end{aligned} \qquad n=0,1,2,\cdots, \tag{5.7}$$

where

$$\begin{aligned} a_{2n+1} &= -(\alpha+n)(\gamma-\beta+n)/(\gamma+2n)(\gamma+2n+1), \\ a_{2n+2} &= -(\beta+n+1)(\gamma-\alpha+n+1)/(\gamma+2n+1)(\gamma+2n+2), \end{aligned} \qquad n=0,1,2,\cdots, \tag{5.8}$$

and consequently we have the correspondence:

$$\frac{F(\alpha,\beta,\gamma;x)}{F(\alpha,\beta+1,\gamma+1;x)} \sim 1 + a_1x/1 + a_2x/1 + a_3x/1 + \cdots . \tag{5.9}$$

*Perron, *loc. cit.*, p. 304.

We mention two contrasting examples which can be derived from (5.9). The first is the correspondence

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
$$\sim 1+x/1-(x/1\cdot 2)/1+(x/2\cdot 3)/1-(x/2\cdot 3)/1+(x/2\cdot 5)/1-\cdots,$$

where the power series has infinite radius of convergence; and the second is the correspondence

$$1+x-1!x^2+2!x^3-3!x^4+\cdots$$
$$\sim 1+x/1+x/1+x/1+2x/1+2x/1+3x/1+\cdots,$$

where the power series has zero radius of convergence. In the first of these the continued fraction converges for every value of x to the function e^x. But the surprising thing is that the second continued fraction also converges for all values of x which are not real and $\leqq 0$, and the function represented is

$$1+x\int_0^\infty \frac{e^{-u}du}{1+xu}, \tag{5.10}$$

which is an analytic function of x when x is not real and $\leqq 0$. What, if any, is the relation between this function and the totally divergent power series from which it arose? The answer is that (5.10) is precisely the *Borel sum* of the divergent series. And the function (5.10) is a solution of the differential equation

$$x^2\frac{dy}{dx}+y=1+x,$$

which the power series satisfies formally! Moreover, the continued fraction may be used to compute values of the solution of this differential equation for particular values of x.

In conclusion we shall state the celebrated Stieltjes moment problem and the solution of this problem which Stieltjes obtained with the aid of continued fractions. Let

$$0<x_1<x_2<x_3<x_4<\cdots$$

be an infinite sequence of points on the real axis and suppose that there is a positive mass m_i concentrated at the point x_i such that the moments of order k:

$$c_k=\sum_{i=1}^\infty x_i^k m_i,\ k=0,1,2,\cdots,$$

are all finite. Under these conditions the determinants

$$\begin{vmatrix} c_0,c_1,\cdots,c_n \\ c_1,c_2,\cdots,c_{n+1} \\ \cdot\quad\cdot\quad\cdot \\ c_n,c_{n+1},\cdots,c_{2n} \end{vmatrix}, \qquad \begin{vmatrix} c_1,c_2,\cdots,c_{n+1} \\ c_2,c_3,\cdots,c_{n+2} \\ \cdot\quad\cdot\quad\cdot \\ c_{n+1},c_{n+2},\cdots,c_{2n+1} \end{vmatrix}$$

are all *positive*. Conversely, let $c_0,c_1,c_2,\cdots$ be any infinite sequence of real numbers for which the above determinants are all positive. Then Stieltjes showed that it is always possible to determine a distribution of mass along the non-negative half of the real axis such that $c_0,c_1,c_2,\cdots$ are the 0th, 1st, 2nd,$\cdots$ order moments of the distribution. To determine whether or not this distribution of mass is *unique*, one constructs the continued fraction corresponding to the power series $1+c_0^x-c_1x^2+c_2x^3-c_3x^4+\cdots$. It turns out that this continued fraction can be written in the form

$$1+x/b_1+x/b_2+x/b_3+x/b_4+\cdots$$

where the b_i are the real *positive* constants. If the series $\sum b_i$ *diverges*, the moment problem has a unique solution, i. e., there is one and only one distribution of mass for which c_k is the kth order moment $(k=0,1,2,\cdots)$; but if the series $\sum b_i$ *converges*, the moment problem has an infinite number of different solutions.

Important among the many applications of the Stieltjes continued fraction is the problem of expanding an arbitrary function in series of of orthogonal polynomials. This application was first made by O. Blumenthal (Göttingen Diss., 1898), and later by Shohat and Sherman, [Pros. Nat. Acad. of Sciences, Vol. 18 (1932), pp. 283-287; also Sherman T. A. M. S. Vol. 35 (1933), pp. 64-87].

Carleman (Les fonctions quasi analytiques, Paris, 1926, Chapter VIII) has applied Stieltjes continued fractions to prove important theorems on quasi-analytic functions.

Hellinger (Math. Annalen Vol. 86 (1922), pp. 18-29) has applied the theory to the study of infinite systems of linear equations.

An important generalization of the moment problem was undertaken by E. B. Van Vleck [T. A. M. S. Vol. 4 (1903), pp. 297-332], in which he permitted a distribution of mass along the entire real axis. This problem was later completely solved by H. Hamburger (Mathematische Annalen, Vols. 81, 82). Other names associated with this problem are Nevanlinna, Riesz, Hausdorff, and Schoenberg.

It has not been the purpose of the authors to give a complete bibliography. A bibliography which is fairly complete up to 1929 will be found in Perron's book.

Humanism and History of Mathematics

Edited by
G. Waldo Dunnington

The Nature of Mathematics

By A. L. O'Toole
Chicago, Illinois

What is mathematics? Is it reality? Absolute truth? An exact science? And what has been its rôle in human thought? When the writer attempted to discuss this topic a few years ago he was reminded of what he once heard a well known professor say. He said that if he asked a freshman a question he usually got an answer immediately. A year or two later the same student would hesitate a moment before answering. By the time he was a senior he perhaps would admit that he just didn't know the answer. Twenty years ago the writer probably would have thought it an easy question to answer if he had been asked what is mathematics. Today he must admit that he isn't sure just what mathematics is. However, there are several things which he thinks are not mathematics and which mathematics is not.

He is not ready to admit that mathematics is the science of quantity and space, for mathematics deals with relationships which are neither quantity nor space. He is not willing to admit that it is the science of numbers, for the study of numbers is only part of mathematics. It is not the presentation in serviceable form of a body of useful information, for there is no one so wise that he can tell what will prove to be useful. Important parts of mathematics, such as conic sections, were studied for centuries before anyone became aware of their practical significance. The ancient Greeks were a theoretical people, intensely interested in abstract science and the world owes many mathematical developments of greatest utility to them. The practical-minded Romans took control of the western world from the Greeks and there followed a period of nearly two thousand years devoid of great mathematical contributions. The Romans were not the dreamers that the Greeks had been.

Mathematics is not a body of propositions logically connected, nor the science which draws necessary conclusions; it is not pure

logic, nor is logic pure mathematics. Many of the great developments in mathematics have been neither deductive nor necessary, as for example, the invention of coordinate geometry by Descartes in 1637. Mathematics and logic are as distinct as mathematics and physics. One is a powerful aid to the other and that is all. If mathematics is logic then all science is logic, and if logic is mathematics then all science is mathematics. One has but to study the history of mathematics during the past two hundred years to see that the logical rigor of one generation has been shattered by the next, without destroying the results illustrious predecessors were able to obtain even though they lived in what has been called a fool's paradise. Their intuition was far ahead of their logic.

Nor does the writer agree that everything in mathematics is based on intuition of the natural numbers $1,2,3,\cdots,$—and that much of mathematics is "just talk". The irrational numbers and the imaginary numbers are not products of the intuition. The child learns from experience rather than from logic or intuition that $2+2=4$. In teaching elementary mathematics to the child one's attitude is somewhat like that of the Bellman who said "What I tell you three times is true". Likewise, common sense is not lord and master over mathematics; for common sense usually frowns on originality, and ingenious ideas are sometimes thought to be nonsense simply because they are new.

It has been said that the intellect is engaged in an endless search for truth and that it is only completely satisfied when it has attained absolute truth. But where does one find absolute truth? In the natural sciences many of our beliefs stand or fall with certain hypotheses such as the Ether. And so perhaps one is inclined to have recourse to mathematics and say that here at any rate is absolute truth, and plenty of it. If there is an exact science, surely it is mathematics. Isn't it absolute truth that $2+2=4$? Isn't it absolute truth that the three angles of a triangle together make an angle of 180°? It might make some of the work in mathematics simpler if these were absolute truths. But they aren't absolute truths. All one can say for them is that for thousands of years no one thought of doubting their absolute truth. And if they were absolute truths non-Euclidean geometry and many other interesting developments of modern mathematics would have been impossible. $2+2=4$ depends upon the definitions of number and operation, and the other depends upon the parallel axiom. And so the exactness of mathematics has been said to be as elusive as the ghost in Hamlet.

So much for what mathematics is not. Now perhaps it will be possible to get some impression of what mathematics really is.

Mathematics has been called the greatest and most original invention of the human mind. The writer prefers to think of it as one of the great social institutions built up by cooperation during the long history of civilization,—a social institution second only to language, and a very close second at that; for it may be thought of as a part of language. Like language it is a mode of thinking, and necessary for the evolution of the human race. Number, for instance, is not an instinct. It is a device worked out by the race for the purpose of making possible the exactness of thought and expression essential to cooperative living.

The chief characteristic of mathematics is its desire for generalization, for complete abstraction. And it should be remembered that the ability of the human mind to attain high levels of generalization and abstraction is its most distinctive feature. Generality has been called the soul of mathematics. When generality is lacking the work is often said to be unscientific. The geometry of Euclid has been criticized for its lack of generality because a different method is used in each of the elementary theorems. The mathematician tries to avoid restrictions and limitations on his theorems. It has been said that Laplace dreamed of a single formula that would encompass the whole universe and all its potentialities. The development of symbolism in mathematics is in keeping with this desire for generality and it is without doubt true that much of the scientific progress of recent centuries is the result of the adoption of the Arabic numerals, digital notation, and the like.

The mathematicians of the world today may be divided into three schools: the logistic school, the intuitionist school and the axiomatic school. The logistic school, headed by Peano in Italy, and Russell and Whitehead in England and America, maintains that mathematics was discovered from logic, or that mathematics is a part of logic. This school reduces all mathematics to symbolic logic and has developed a sign language which few can read. The intuitionist school, headed by Brouwer of the Netherlands, takes the opposite view and holds that logic rests on mathematics and hence that logic cannot be the foundation of mathematics. The intuitionist says that it is nonsense to give a definition without a method of applying it. For instance, although one can prove that of two rational numbers the first is less than, equal to, or greater than the second, one cannot demonstrate this for numbers in general. Hence the intuitionist says that $x<y$, $x=y$ and $x>y$ are not mutually exclusive. The intuitionist solves the problems of mathematical existence on the basis of constructibility. This takes away many *reductio ad absurdum* proofs and much of the

discussion of the infinite, for they say that the law of the excluded middle is applicable only to finite sets. The axiomatic school, headed by Hilbert of Germany, argues with the logistic school over the possibility of admitting to mathematics anything that may lead to contradiction or paradox. Hilbert has done the best work in geometry that the world has ever known but there is as yet no system which guarantees the impossibility of paradoxes. Hilbert believes in the existence of the infinite, a sort of growing infinite rather than completed infinite. For him mathematics is not part of logic. But the sign language of the logistic school is perfectly adaptable to Hilbert's work.

It should now be clear that no matter to which school of mathematics one belongs, logic is his most powerful aid, and at the same time his most infallible criterion of exactness. In principle, mathematics is good or exact if the conclusions reached are logically obtained from the hypotheses stated or implied, regardless of the truth of the hypotheses themselves. But since there is as yet no perfect system in mathematics, exactness exists very often in the aim rather than in the achievement.

In reaching logical conclusions the mathematician has at his disposal both deduction and induction. But since deduction is by far the simpler of the two processes, it is not surprising that most mathematics is deductive and that most mathematicians are trained in deductive mathematics exclusively. In fact some of the greatest mathematicians claim that logical reasoning is necessarily deductive and that there is no room at all for induction in mathematics. However, some of the workers in mathematical statistics and physics feel compelled to recognize the necessity of inductive inference and profess to have found means of accomplishing logical results by the inductive process, in which such things as weight of evidence, measures of rational belief, uncertain inference, probability and likelihood are involved.

Thus it would seem that the mathematician may use something from each of the three schools. To begin with he has something from his intuition, though certainly not the natural numbers. There is no instinct which would lead a man to write the Arabic numerals, and besides, they were unknown in western Europe until the twelfth century. A mathematician is born with the equipment necessary for abstract thinking. Soon he acquires a sense of quantity, of any and some, of more and less, and a sense of form. Then the notions of order, group, correspondence, periodicity and perhaps some sort of infinity begin to develop in him.

Much of modern mathematics could not have been discovered through the senses, being entirely unobvious. The importance of

logic has been mentioned already. The use of axioms as a method of unification is very desirable in mathematical generalization. Even the symbolism of the logistics may be helpful. A major advantage of the axiomatic procedure is that its precision is an aid to clear thinking. Another is that the use of words devoid of content encourages experimentation with new combinations of hypotheses, and thus may lead to new discoveries and the possibility of wide practical applications. Obviously consistency of the axioms is of very great importance. But, on the other hand, freedom from contradiction does not in itself constitute mathematics. To tell one he must not do so and so in proving a mathematical proposition, because someone may later discover a paradox that will ruin the axioms, is as nonsensical as to tell one to stay away from school today because a motor car may run over him on the way. The same applies to the use of the law of the excluded middle.

It has been pointed out already that elementary mathematics evolved from the exigencies of people living together on a basis of cooperation. The higher branches of the subject constitute one of the noblest of the arts developed by the human race. Certainly there would be no such civilization as we enjoy today were it not for mathematics. Recorded history shows only two significant periods of intellectual progress. The first began about the time of Pythagoras in the sixth century B. C. and ended about the time of Archimedes in the third century B. C. It was a period of great progress in mathematics. There followed a long period in which there was little progress, intellectual or mathematical. A new era began about the middle of the seventeenth century with the momentous mathematical developments introduced by Descartes and others. Wide intellectual and scientific advance has been the result. And the end is not yet. Man's vision of the atom and of the universe and of man himself have been altered. And all of the great advances depend upon mathematics for their ultimate answers. At the present time the chief difficulty in mathematics seems to be the notion of the infinite. Some mathematicians hope that in the not too distant future the idea of the infinite will be made clear and that then much greater exactness in mathematics will be possible and mathematics will be a more powerful tool of the scientist. The writer doesn't believe that the idea of the infinite will ever be made clear. The next best thing then would be an agreement among mathematicians and philosophers about the infinite.

The books and articles which the writer has found to be most helpful with regard to mathematical philosophy are listed in the bibliography. There is much interesting reading in them.

BIBLIOGRAPHY

BELL, E. T.—*The Queen of the Sciences*, Baltimore, The Williams and Wilkins Co., 1931.

BROUWER, L. E. J.—*Intuition and Formalism*, translated by Arnold Dresden, Bull. Amer. Math. Soc., 20 (1913), pp. 81-96.

DRESDEN, A.—*Brouwer's Contribution to the Foundations of Mathematics*, Bull. Amer. Math. Soc., 30 (1924), pp. 31-40.

EINSTEIN, ALBERT—*Physics and Reality*, translated by Jean Picard, Journal of the Franklin Institute, 221 (1936), pp. 349-382.

FISHER, R. A.—*The Logic of Inductive Inference*, Jour. Roy. Stat. Soc., 98 (1935), pp. 39-54.

HILBERT, DAVID—*The Foundations of Geometry*, translated by E. J. Townsend, Chicago, The Open Court Pub. Co., 1902.

JUDD, C. H.—*The Psychology of Social Institutions*, New York, The Macmillan Co., 1926.

KEYSER, C. J.—*The Pastures of Wonder*, New York, Columbia University Press, 1929.

PIERPONT, JAMES—*Mathematical Rigor, Past and Present*, Bull. Amer. Math. Soc., 34 (1928), pp. 23-53.

RUSSELL, BERTRAND—*Introduction to Mathematical Philosophy*, New York, The Macmillan Co., 1919.

WEYL, HERMANN—*Philosophie Der Mathematik und Naturwissenschaft*, München und Berlin, R. Oldenbourg, 1927.

WHITEHEAD, A. N.—*An Introduction to Mathematics*, New York, H. Holt and Co., 1911.

WHITEHEAD, A. N.—*Science and the Modern World*, New York, The Macmillan Co., 1925.

WHITEHEAD, A. N. and RUSSELL, BERTRAND—*Principia Mathematica*, 3 Vols., Cambridge, University Press, 1925.

YOUNG, J. W. A.—*Monographs on Topics of Modern Mathematics*, New York, Longmans, Green and Co., 1932.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Why Learn Mathematics?*

By EDWIN G. OLDS
Carnegie Institute of Technology

It is futile for any one person to attempt to give a comprehensive answer to so broad a question. First we need to recognize that any answer is a function of several variables, namely: the definition of "learn" as to quality, the specification of "mathematics" as to type, and, particularly, the needs of the person for whom the answer is given. Second, we need to decide whether the answer is to be a superficial one, couched in terms of salesmanship platitudes and implications, or whether we have enough interest in the function to study it with care in the various regions of its definition. And, third, we should try to discover proper applications for our findings.

All of us are quite sure that we know what it means to "learn" mathematics, but I doubt the possibility that we have any precise agreement as to the connotation of the word "learn". Perhaps you will agree that after a person "learns", he "knows". However, that doesn't help much because we have no better agreement on the meaning of "know", or "understanding", or "appreciate". This fact becomes most apparent when the college teacher states that the freshman doesn't know algebra, in spite of the fact that the high-school teacher has given grades to certify the possession of knowledge. Likewise the statement by the grade-school teacher that arithmetic is known is often questioned. Furthermore, there is some suspicion that many teachers, college, high-school, grade, do not know arithmetic, algebra, calculus. Perhaps it is not fitting to dwell longer on this painful subject but the observation might be made that, while ignorance, like poverty, is no disgrace, it is quite unnecessary for a young and normal person in a land of plenteous opportunities. Certainly learning is something more than accumulating a list of facts like beads on a

*A talk delivered at a meeting of the Mathematics Teachers Association of Western Pennsylvania, February 25, 1939, at Pittsburgh, Pa.

string; it is more than the ability to arrange the facts in intelligent patterns. Perhaps it is better compared to the establishment of interconnections like the intricate wiring of a submarine, such that the throwing of a switch sets in operation a chain of adjustments which combine to produce a desired result.

It might seem easier to define mathematics, but here we are foiled again unless we fall back on definition by illustration and give arithmetic, algebra, geometry as examples. Some people would define mathematics as a tool for measurement, others as a mode of thought by which logical conclusions can be reached. Euler has called it the Queen of the Sciences; some of our students speak of it as the bane of their existence. When you have worked out a good definition for mathematics, test it by seeing whether it separates mathematics and physics, or mathematics and logic. In many institutions of learning, business arithmetic, accounting, mechanics, descriptive geometry, statistics, astronomy are not classified with mathematics. If you explain that these subjects are applied mathematics, you expose yourselves to the question as to what you are applying. Also you bear the burden of proving that subjects traditionally classified as pure mathematics, such as plane geometry, escape the stigma of being called "applied". So mathematics is rather elusive and ethereal when we try to pin it down.

Let us turn to a consideration of our third independent variable, the needs of the person for whom the answer is to be given. Most of you have studied some mathematics and, in the eyes of the world, at least, have "learned" mathematics. If you ask as to why you should have learned mathematics the question is much easier to answer than it would have been when you began to learn to count, or when you started geometry, or when you decided to major in mathematics in your college course. We can count results now which we could not have guaranteed then. In the first place you are making a rather comfortable living by teaching what you have learned; second, you have derived a certain amount of satisfaction from the acquisition of a body of knowledge which you can contemplate as a thing of beauty and can utilize for your entertainment and for the acquisition of other related experiences; third, you have had the advantage of possessing a powerful tool for daily living which has helped you to understand better and cope with the physical world into which you were born, has assisted you in your determined effort to stay in that world as long as possible, and has added to your capacity to contribute your share to the progress of civilization. So, on the whole, we may conclude that it was a good thing for you to learn mathematics. And, since the past seems to

provide the safest guide for the future it seems satisfactory to predict that those students of ours who plan to teach mathematics will do well to devote a large share of their attention to learning mathematics. How large this share should be is a question of much difference of opinion, as is apparent when we compare the requirements for state certification with the recommendations made by the Commission on the Training and Utilization of Advanced Students of Mathematics. (*Am. Math. Monthly*, Vol. 42, 1935, pp. 263-277).

Since the answer to our question seems much too easy when we consider only prospective teachers of mathematics, let us focus our attention on a domain where the answer is much harder to obtain. Namely the general case of the student who is in the threshhold of exposure to some educative process. At any point, from the time he is born until the time that he leaves this vale of tears, the child and his sponsors have a right to ask the question, "What mathematics should I study, and why?" Now you see that the answer becomes very hard to give because it is so involved with the answers to a number of other questions, yet we must, at least in general terms, be ready to provide a satisfactory answer at every point of time from the cradle to the grave. Of course, by now, you have probably guessed the truth that this paper will provide no answer. It is designed, only, to guide your thoughts along paths where possible answers may lie, leaving to you the choice of satisfactory conclusions, reached on the basis of your own moral, political and educational philosophies.

Here in America we are committed to the principle that a democratic form of government depends for its existence on the eduation of its citizenry. We have inherited the tradition that every individual is entitled to as much formal education as we can afford to give him. The history of our public school system shows that we have tended to extend this education in both directions as rapidly as our resources would allow it; we get the child earlier and keep him later. At the same time we have been extending the length we have been increasing the breadth, and, perhaps, decreasing the depth. A comparison between the list of subjects taught fifty years ago and bountiful curricular offering today reveals the extent to which we have changed our educational views. I need not remind you that mathematics has suffered. Its position has been stripped of much of the fortification of tradition. Along with the classics, it finds itself no longer fashionable. It does not follow, however, that it is no longer necessary, in greater or less degree, for the normally intelligent portion of our population. It does mean, however, that at any stage of the game, mathematics must compete, on an equal basis, with a great variety of other interesting and important subjects.

If we could foresee the destiny of our students it would simplify our work in helping them with their choice of course and subjects. While it is not possible to predict with certainty the life work of any individual, research in the field of vocational training and psychological analyses of skills and aptitudes are proceeding in the direction of providing a scientific basis for indicating the general field of endeavor in which the student will have the best chance of success. Already we known, in a general way, the characteristics necessary for a good engineer or artist or business executive; we know that persons of a certain level of intelligence have a very poor chance in college, while persons at higher levels have better and better chances in university and graduate work. It is not too much to hope that the time will come when we can test the kindergartner and predict with a high degree of accuracy the manner in which he will earn his living. However, for the present, we must stumble along as best we can. At the elementary level we have succeeded in reducing arithmetic to the point where the remainder contains only those essentials which appear to be the minimum for any human being who is to live and work in our present social order. Of course, the accomplishment of this success could not help but remove much which would prove of considerable benefit to an important part of our school population. Beyond the first six grades, we are still in the process of study and experimentation.

In considering mathematics or anything else it is well to keep in mind the privilege of the individual to rise above the dead level of mediocrity and contribute significantly to the world's well-being. Now, as always, we need frontier thinkers; men who are trained to lead in religion, in science, in government and in business. He who wins support for a weakened curriculum by prating of equal opportunities for all may be sacrificing future benefits for present comfort. The easiest and most expedient course of action may not be the best or most courageous. Any program which does not provide each individual with the opportunity to develop himself with the maximum of efficiency of time and effort not only is unfair to the individual but also robs society. Probably friends of mathematics have lost ground by making the mistake of failing to admit that many people are able to lead happy and fruitful lives with a very meagre knowledge of mathematics. On the other hand there is a suspicion that the inherent difficulty of mathematics has been the concealed reason for its disappearance from many a course of study. If, during the past twenty years, teachers of mathematics had been willing to lower standards ahead of orders instead of after them, it is a question as to whether so many good arguments would have been discovered as to its general inutility.

In answering the question, "Why learn mathematics?" it is not necessary to depend on one's individual experience and observations. The thoughts and findings of others on the subject have received a liberal amount of space in recent books and magazines. At the close of this article, there are listed a few of the papers, bearing on the subject, which have appeared recently in three magazines which devote more or less space to questions of mathematical pedagogy. These papers present diverse, and sometimes conflicting, views, but they should serve as an aid to provoke thought and a guide to sound conclusions on a question we all must answer somehow, if we are to be worth our salt as mathematics teachers.

Finally, in assessing the value of mathematics to any individual it is worth while to consider the thesis that the success of democratic society depends on mutual understanding and that mutual understanding depends on the adoption of common units of measure. Let us examine the significance of this statement. It seems to be obvious that fundamental principles must be expressible in terms which have the same meaning for all. The terms which have most universal similarity of interpretation are those concerned with measurement of quantity. Whenever we begin to content ourselves with qualitative expressions we run the risk of great difference of interpretation. Qualitative terms are interpreted by the individual on the basis of his own experience, while quantitative terms carry precision of connotation, in case the individual possesses a modicum of mathematical training.

Let me illustrate this by two simple examples: In a manual for the identification of species of rabbits, I find the statement that a certain bone is short and broad. Knowing nothing about the normal dimensions of this particular bone, the description provides little information. If, however, the book had stated that the bone was three inches long, two inches wide, and one-half inch thick it would have removed the possibility that the author and I would differ in our classification. As a second example consider the attempts to obtain mutual agreement on what is meant by liberalism. Try to classify your friends on this score; get a friend to do the same; compare results; try to arrive at a mutually satisfactory definition; now examine your processes and observe how esssentially mathematical they have been. Some one has said that branches of learning become scientific only as their elements are capable of measurement. Likewise we might say that ideas become capable of being communicated or received only to the extent that they allow quantitative expression.

In summary, we have indicated that the answer to the question, "Why learn mathematics?" is a function of the definition of "learn",

the type of "mathematics", and the needs of the individual. The nature of the set of possible values for each of these three variables has been partially examined. In particular, we have suggested certain important regions of definition for the third variable. In so doing we have implied the existence of certain possible values for the function. However, this paper does not pretend to have contributed any considerable information about the function, but rather to have offered general suggestions for its investigation. In other words, the author has not deprived you of the pleasure of investigating for yourselves the answer to the question, "Why learn mathematics?"

SUGGESTED READING ON THE QUESTION, "WHY LEARN MATHEMATICS ?"

The Mathematics Teacher, published at Menasha, Wis. and New York, N. Y., by the National Council of Teachers of Mathematics.

VOLUME 29 (1936)

MEEK, A. R.—Recreational aspects of mathematics in the junior high school, pp. 20-22.

ROSANDER, A. C.—Quantitative thinking on the secondary school level, pp. 61-66.

HEDRICK, E. R.—Crises in economics, education, and mathematics, pp. 109-114.

POTTER, M. A.—The human side of mathematics, pp. 123-128.

HOTELLING, H.—Some little known applications of mathematics, pp. 157-169.

WILLIAMS, K. P.—Why we teach mathematics, pp. 271-280.

MOULTON, E. J.—Mathematics on the offense, pp. 281-286.

MUNCH, H. F.—A brief professional philosophy for teaching of high school mathematics, pp. 334-339.

NOYES, D.—Finding social mathematics in school activities, pp. 340-345.

WRINKLE, W. L.—Mathematics in the modern curriculum for secondary education, pp. 374-380.

KINNEY, L. B.—The social-civic contribution of business mathematics, pp. 381-386.

VOLUME 30 (1937)

SPRAGUE, J. B.—Some interesting detours in algebra, pp. 29-30.

MALLORY, U. S.—Providing for individual needs in mathematics, pp. 214-220.

GRIFFIN, F. L.—Mathematics, a tool subject or a system of thought? pp. 223-228.

LEISENRING, K. B.—Geometry and life, pp. 331-335.

OLDS, E. G.—Let's check the hypothesis, pp. 358-359.

VOLUME 31 (1938)

RITER, H. E.—The enrichment of the mathematics course, pp. 3-6.

BUCKINGHAM, B. R.—Significance, meaning, insight—these three, pp. 24-30.

SEIDLIN, J.—What shall we do with our unfit? pp. 75-77.

KEMPNER, A.—On the need of cooperation between high school and college teachers of mathematics, pp. 117-123.

HARTUNG, M. L.—Some problems in evaluation, pp. 175-182.

MADDOX, A. C.—The expediency of compromise in mathematical curricular and instruction, pp. 259-263.

BRESLICH, E. R.—The nature and place of objectives in teaching geometry, pp. 307-315.

HALL, E. L.—Applying geometric methods of thinking to life situations, pp. 379-384.

The American Mathematical Monthly, published at Menasha, Wis. and Evanston, Illinois, by the Mathematical Association of America.

DRESDEN, A.—A program for mathematics, Vol. 42 (1935), pp. 198-208.

LANGER, R. E.—The new mathematics "requirement" at the University of Wisconsin, Vol. 42 (1935), pp. 208-212

DUDLEY, A. M.—The type of mathematical training needed by electrical engineers, Vol. 42 (1935), pp. 301-306.

SCHAAF, W. L.—Required mathematics in a liberal arts college, Vol. 44 (1937), pp. 445-453.

NATIONAL MATHEMATICS MAGAZINE, published at Baton Rouge, La., by Louisiana State University:

MILLER, W. M.—A discussion of the methods of science, history, art and mathematics, Vol. 10 (1936), pp. 200-204.

SEIDLIN, J.—The place of mathematics and its teaching in the schools of this country, Vol. 10 (1936), pp. 304-307.

SANDERS, S. T.—Quality and mathematics, Vol. 11 (1937), p. 348.

EDINGTON, W. E.—Vitalizing mathematics, Vol. 12 (1938) pp. 27-38.

BLUMBERG, H.—What is essential in teaching mathematics? Vol. 12 (1938), pp. 393-402.

The place of mathematics in modern education, The Eleventh Year book of the National Council of Teachers of Mathematics, Bureau of Publications, Teachers College, Columbia University, New York City.

The place of mathematics in secondary education, A Preliminary Report by The Joint Commission of The Mathematical Association of America, Inc., and The National Council of Teachers of Mathematics, Edwards Brothers, Inc., Ann Arbor, Michigan, 1938.

Mathematical World News

Edited by
L. J Adams

The 1939 meeting of the Texas Section of the Mathematical Association of America was held in Abilene on March 31 and April 1, Hardin-Simmons, McMurray College and Abilene Christian College acting jointly as hosts. Professors Michie, Sparks, Heineman, Langston, and Dr. Brady of the department of mathematics of Texas Technological College attended the meeting.

Dr. R. S. Underwood and Professor J. N. Michie, University of Texas, will attend the dedication of the McDonald Observatory at Fort Davis, Texas to be held from May 5 to 8. It will be conducted jointly by the University of Chicago and the University of Texas. On this occasion an astronomical symposium will be held under the auspices of the observatory and of the Warner and Swasey Company, under the general title *Galactic and Extragalactic Structure.* Those wishing to attend the dedication and the symposium are requested to make hotel reservations by writing to the observatory, Fort Davis, Texas.

The Office National des Universités et Écoles Françaises is offering a fellowship of 18,000 francs for nine months of study at any French university, free transportation to and from France (tourist class, French Line), and railroad fare from Le Havre to Paris and return to advanced students who have specialized in science, preferably men who have already obtained doctorates in mathematics, physical science, chemistry or biology. The Société des Amis de l'Université de Paris offers a fellowship of 18,000 francs to graduate students to undertake scientific research in Paris. The fellowships are open to men only. The candidate must:

(a) Be American born;
(b) At the time of application be a graduate student in a university or professional school of recognized standing; or an instructor who has completed his graduate work and wishes to study abroad;
(c) Be of a good moral character and intellectual ability, and of suitable personal qualities;

(d) Present a certificate of good health;
(e) Possess ability to do independent study and research;
(f) Have a practical ability to do independent study and research in general subjects and in his own special field, and be able to speak French and understand lectures delivered in French.

Preference will be given to applicants between the ages of 25 and 35. For application forms address the Student Bureau, Institute of International Education, 2 West 45th Street, New York, sending ten cents to cover costs. The closing date for filing applications with completecredentialsis April 1, 1939. It is regretted that this news came too late for insertion in the previous issue of the Magazine.

Under the direction of the department of mathematics a conference on the Calculus of Variations will be held at the University of Chicago from June 27 to June 30, with a number of invited addresses by members of the faculty of this and other universities on topics connected with the calculus of variations in which the speakers have been especially interested.

Dr. D. W. Hall has been appointed to the staff of the mathematics department of Brown University. Dr. Hall is a graduate of the University of Virginia, was a member of the department of mathematics at the University of California at Los Angeles for the academic year 1937-38, and at the present time is National Research Fellow at the University of Pennsylvania.

Mr. Gordon Mirick, Lincoln School of Teachers College (Columbia University), is preparing a combined course in algebra and geometry for use in high schools. It is designed to replace the separate year courses now offered.

The following courses in mathematics are announced for the summer quarter of 1939:

Texas Technological College, Lubbock, Texas, First term June 5 to July 14; Second term July 17 to August 24. In addition to the usual elementary courses, the following advanced courses will be offered: By Associate Professor Earl L. Thompson: Modern Higher Algebra (first term); Mathematics of Finance (first term). By Professor R. S. Underwood: Elementary Theory of Numbers (second term). By Professor J. N. Michie: Advanced Algebra (first term); Reading and Research Course (second term); Differential Calculus (first term); Integral Calculus (second term). By Professor Fred W. Sparks: Applied Calculus (first term); Reading and Research Course (first term).

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, College Park, Md.

SOLUTIONS

No. 255. Proposed by *V. Thébault*, Le Mans, France.

An arbitrary point M is taken within a square P. The projections of M upon the sides of the square are the vertices of a quadrangle P_1, called the first pedal of M. The projections of M upon the sides of P_1 form the quadrangle P_2, the second pedal of M.

(1) Show that every fourth pedal, P_{4n}, is again a square.
(2) Calculate the sum of the areas of P, P_4, $P_8, \cdots$.
(3) What are the angles made by the homologous sides of P and P_4, P_4 and P_8, etc?.

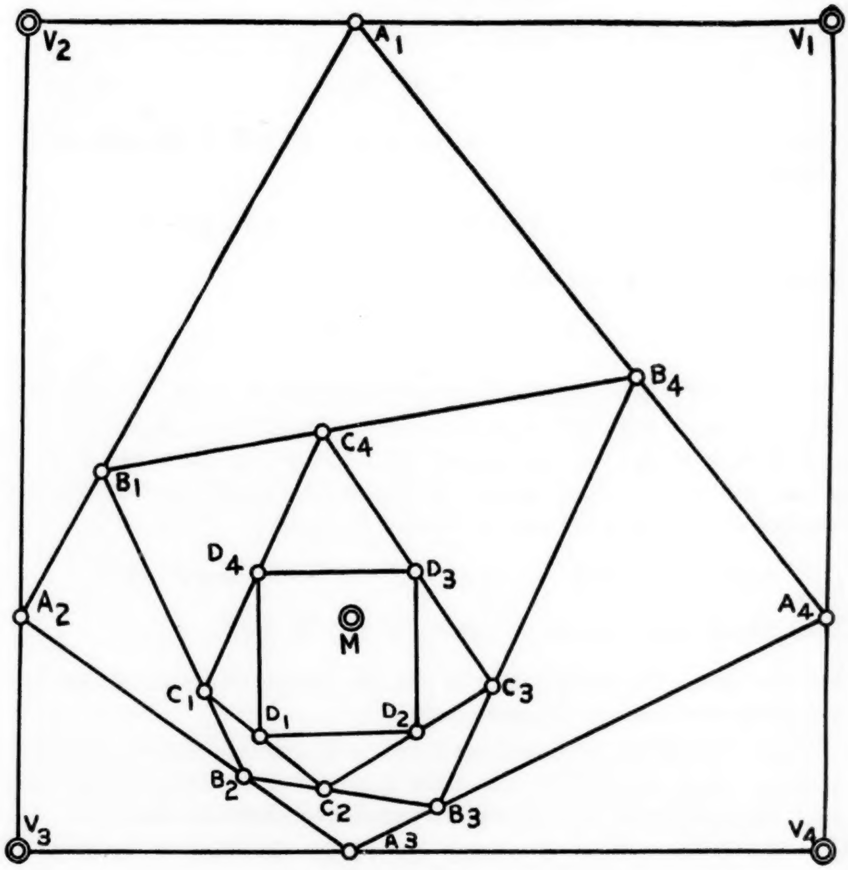
Solution by *Walter B. Clarke*, San Jose, California.

Let successive vertices of the square P be V_i, and those of P_1, P_2, P_3, P_4 be A_i, B_i, C_i, D_i, respectively, with $i=1,2,3,4$. Let $V_1A_1=a$, $V_2A_1=b$, $V_2A_2=c$, $V_3A_2=d$.

Part I: Obviously, $A_1A_2=\sqrt{(b^2+c^2)}$ and $A_2A_3=\sqrt{(b^2+d^2)}$. Also $B_1A_2=b^2/\sqrt{(b^2+c^2)}$, by right triangle A_1MA_2; $B_2A_2=b^2/\sqrt{(b^2+d^2)}$, by right triangle A_2MA_3. Whence

$$B_1A_2/A_2B_2=A_2A_3/A_1A_2.$$

Thus B_1B_2 is antiparallel to A_1A_3 with reference to angle $A_1A_2A_3$. This relation holds for the four sides of quadrilateral B_i and gives a set of eight angles. By using complementary angles at various points and the fact that the twelve quadrilaterals are cyclic, every angle in the figure is found to be one of the eight.



Each corner of P_4 is found to be composed of a different pair of the eight angles with none repeated. Each pair forms a right angle at A_2 and A_4 (also at A_1 and A_3) so P_4 is a rectangle.

By a somewhat tedious computation along usual lines, working from P inward, it is finally found that

$$D_1D_2=(a+b)/\sqrt{K}, \qquad D_2D_3=(c+d)/\sqrt{K},$$

where $K=(a^2+c^2)(a^2+d^2)(b^2+c^2)(b^2+d^2)/a^2b^2c^2d^2$.

Now, since $a+b=c+d$, the rectangle P_4 is a square. The same process repeated will establish that P_{4n} are squares, but for sections (2) and (3) it is necessary to show that M is similarly placed in P and P_4: Draw MV_1 which at V_1 forms angles Θ and Φ. These same angles will be found at D_1 formed there by MD_1. This holds for $D_{2,3,4}$ and if P_4 be rotated on M until MD_1 lies on MV_1 the two squares will be homothetic.

Part II: It is clear that

$$T = K \cdot T_4,$$

where T_4 represents the area of the square P_4 and T the area of P. The sum of the areas is:

$$\sum T_{4n} = T \cdot (1/K + 1/K^2 + 1/K^3 + \cdots) = T/(K-1).$$

If the area of P be included, the entire sum is

$$KT/(K-1), \qquad (K>1).$$

Part III: The angle formed by A_1A_2 and V_1V_2 is $\alpha = \text{arc tan } (c/b)$; that formed by B_1B_2 and A_1A_2 is $\beta = \text{arc tan } (b/d)$; that formed by C_1C_2 and B_1B_2 is $\gamma = \text{arc tan } (d/a)$; and that formed by D_1D_2 and C_1C_2 is $\delta = \text{arc tan } (a/c)$. Using $\tan(\alpha+\beta)$ and $\tan(\gamma+\delta)$, we may obtain the tangent of the angle between V_1V_2 and D_1D_2 as:

$$(cd-ab)(c-d)(b-a)/[(cd-ab)^2 + cd(a^2+b^2) + ab(c^2+d^2)],$$

from which some interesting inferences may be drawn:

1. Since the denominator is always positive, the sign of the tangent depends only on the numerator.
2. The tangent will vanish if any of the factors of the numerator is zero. That is, if M lies on either diagonal or on either medial line of P, the sides of P and P_4 will be parallel. Otherwise not.
3. If $(c-d)$ and $(b-a)$ are comparatively small (i. e., if M is near the center of P) the sides of the squares will be very nearly parallel.

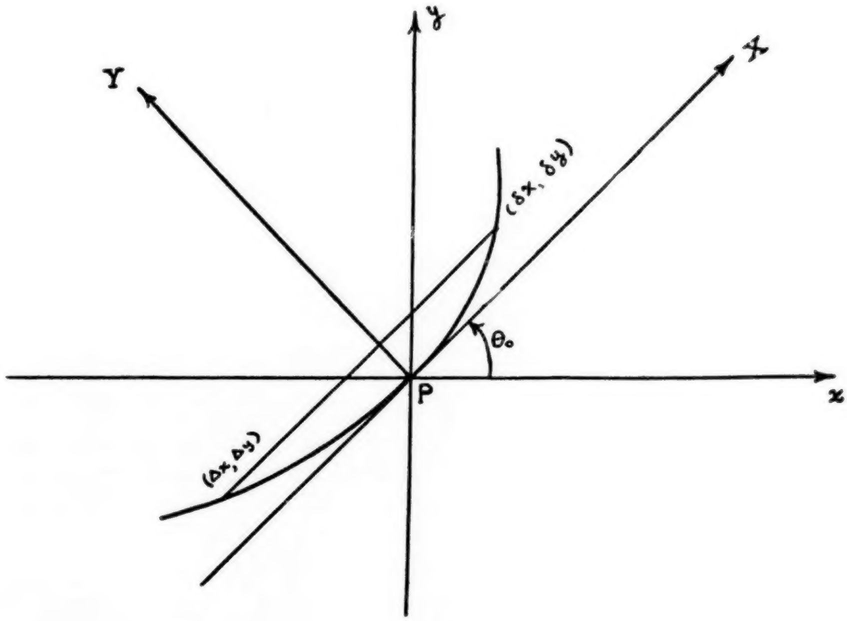
Also solved by the *Proposer.*

No. 256. Proposed by *C. E. Springer*, University of Oklahoma.

Given a plane curve with the equation $y = y(x)$, regular in the neighborhood of the point $P(x_0, y_0)$, and a chord with end-points $A(x_0+dx, y_0+dy)$ and $B(x_0+\delta x, y_0+\delta y)$, parallel to the tangent to the curve at P. If the distance between the chord and tangent is ϵ, prove that

$$\lim_{\epsilon \to 0} \frac{dx+\delta x}{\epsilon} = \pm(\tfrac{2}{3}) \frac{(d^3y/dx^3)\sqrt{1+(dy/dx)^2}}{(d^2y/dx^2)^2}.$$

Solution by *W. E. Byrne*, Virginia Military Institute.



There is no loss in generality if the point $P(x_0, y_0)$ is taken at the origin of coordinates. Consider an auxiliary system of rectangular axes PX, PY with PY oriented in the same manner as the interior normal at P.

The figure is drawn for the case $f''(0)>0$. From

$$(1)\qquad \begin{cases} x = X\cos\Theta_0 - Y\sin\Theta_0 \\ y = X\sin\Theta_0 + Y\cos\Theta_0 \end{cases}$$

it follows that the desired expression is

$$\frac{\Delta x+\delta x}{Y} = \cos\Theta_0 \frac{(\delta X+\Delta X)}{Y} - \sin\Theta_0 \frac{(\delta Y+\Delta Y)}{Y},$$

so we have to calculate

$$(2)\qquad \lim_{Y\to 0} \frac{\Delta x+\delta x}{Y} = \cos\Theta_0 \lim_{Y\to 0} \frac{\Delta X+\delta X}{Y} - 2\sin\Theta_0.$$

Calculation of $\displaystyle\lim_{Y\to 0} \frac{\Delta X+\delta X}{Y}$:

$$Y = a_2X^2 + a_3X^3 + \cdots, \quad a_2 \neq 0$$

$$X = b_0\sqrt{Y} + b_1 Y + \cdots$$

$$b_0 = \pm\frac{1}{\sqrt{a_2}}, \quad b_1 = -\frac{a_3}{2a_2^2}$$

$$\Delta X = -\frac{1}{\sqrt{a_2}}\sqrt{Y} - \frac{a_3}{2a_2^2}Y + \cdots$$

$$\delta X = \frac{1}{\sqrt{a_2}}\sqrt{Y} - \frac{a_3}{2a_2^2}Y + \cdots$$

$$\lim_{Y\to 0}\frac{\Delta X + \delta X}{Y} = -\frac{a_3}{a_2^2} = -\frac{\frac{1}{6}\left(\frac{d^3Y}{dX^3}\right)_0}{\frac{1}{4}\left(\frac{d^2Y}{dX^2}\right)_0^2} = -\frac{2}{3}\frac{\left(\frac{d^3Y}{dX^3}\right)_0}{\left(\frac{d^2Y}{dX^2}\right)_0^2}$$

Evaluation of $\left(\frac{d^2Y}{dX^2}\right)_0, \quad \left(\frac{d^3Y}{dX^3}\right)_0.$

We may use the inverse of (1)

$$\text{(3)} \qquad \begin{cases} X = x\cos\Theta_0 + y\sin\Theta_0 \\ Y = -x\sin\Theta_0 + y\cos\Theta_0 \end{cases}$$

and treat x as a parameter.

$$\frac{dY}{dX} = \frac{-\sin\Theta_0 + y'\cos\Theta_0}{\cos\Theta_0 + y'\sin\Theta_0}, \quad \left(\frac{dY}{dX}\right)_0 = 0$$

$$\frac{d^2Y}{dX^2} = \frac{y''}{(\cos\Theta_0 + y'\sin\Theta_0)^3}, \quad \left(\frac{d^2Y}{dX^2}\right)_0 = y_0''\cos^3\Theta_0$$

$$\frac{d^3Y}{dX^3} = \frac{y'''}{(\cos\Theta_0 + y'\sin\Theta_0)^4} - \frac{3(y'')^2\sin\Theta_0}{(\cos\Theta_0 + y'\sin\Theta_0)^5}$$

$$\left(\frac{d^3Y}{dX^3}\right)_0 = y_0'''\cos^4\Theta_0 - 3(y_0'')^2\sin\Theta_0\cos^5\Theta_0$$

$$\lim_{Y\to 0}\frac{\Delta x + \delta x}{Y} = -\frac{2}{3}\frac{y_0'''}{(y_0'')^2}\sec\Theta_0$$

$$= -\tfrac{2}{3}\frac{y_0'''}{(y_0'')^2}\sqrt{1+(y_0')^2} \text{ if } y_0''>0$$

$$= \tfrac{2}{3}\frac{y_0'''}{(y_0'')^2}\sqrt{1+(y_0')^2} \text{ if } y_0''<0.$$

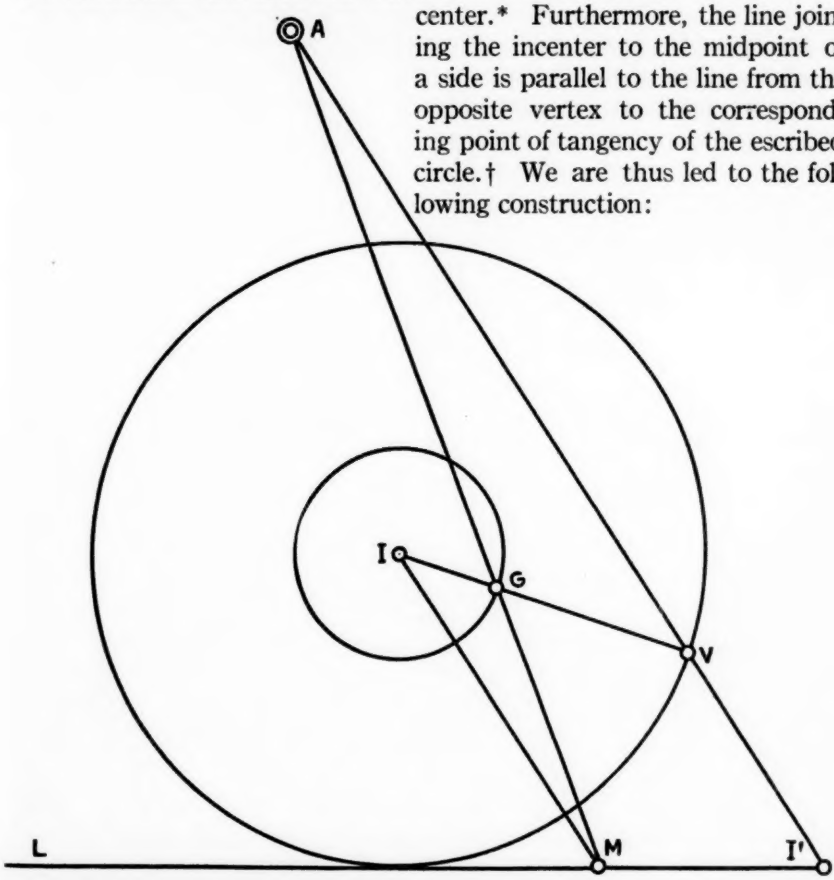
Also solved by the *Proposer*.

No. 259. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle whose verbicenter lies on its incircle.

Solution by *Henry Schroeder*, L. P. I., Ruston, Louisiana.

First, we define the verbicenter as the point of intersection of the lines from the vertices to the points of tangency of the opposite escribed circles. Now it is well known that the distance from the incenter to the centroid is one-third the distance from the incenter to the verbicenter.* Furthermore, the line joining the incenter to the midpoint of a side is parallel to the line from the opposite vertex to the corresponding point of tangency of the escribed circle.† We are thus led to the following construction:



*See Johnson, *Modern Geometry*, p. 225, Sec. 361.
†*Ibid.*

Construct circle with center I and tangent to the line L. With radius equal to one-third of this radius, construct a circle having the same center. From any point M on L draw a line intersecting the smaller circle at G. Draw IG intersecting the larger circle at V. Through V draw a line parallel to IM intersecting MG at A and L at I'. Point A is a vertex of the required triangle, G is the centroid, M is the mid-point of the side a, V is the verbicenter, and I' the corresponding tangent point of the escribed circle.

No. 260. Proposed by *M. S. Robertson*, Rutgers University.

Show that the area of a triangle formed by three tangents to a parabola is one-half the area of the triangle formed by the three points of tangency.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

Let $A=(x_1,y_1)$, $B=(x_2,y_2)$, $C=(x_3,y_3)$ be the points of tangency on the parabola $y^2=x$ and let tangent ECF cut the tangents DA and DB in E and F.

The equations of AD, BD, and EF are

$$2yy_i=x+x_i, \quad (i=1,2,3.)$$

Since the ordinate of the intersection of two tangents equals one-half the sum of the ordinates of the points of contact, we find

$$D=[y_1y_2,\ (y_1+y_2)/2],$$

$$E=[y_1y_3,\ (y_1+y_3)/2],$$

$$F=[y_2y_3,\ (y_2+y_3)/2].$$

Using the determinant form for the area, we obtain*

$$\Delta DEF=[y_1^2(y_2-y_3)-y_2^2(y_1-y_3)+y_3^2(y_1-y_2)]/4$$

and $$\Delta ABC=[y_1^2(y_2-y_3)-y_2^2(y_1-y_3)+y_3^2(y_1-y_2)]/2.$$

Hence $$2\Delta DEF=\Delta ABC.$$

The Encyclopédie des sciences mathématiques, tome III, vol. 3, fasicule 1, p. 118 ascribes this theorem to *A. F. Möbius*, Der barycentrische Calcul, p. 232, 393 or his works I, p. 211, 339. Reference is also given to *D. F. Gregory*, Cambridge Mathematical Journal 2, (1839-41), p. 16/7 (1839).

*Or, more simply, $(y_1-y_2)(y_2-y_3)(y_3-y_1)/4$.—ED.

The following references were supplied by *A. E. Gault* and *John Hallett*:

Loney: Coordinate Geometry, Macmillan (1896) p. 200,
Tanner and *Allen*: Analytic Geometry, p. 230, Ex. 8.

Editor's Note: The following lemma would be found useful here:*

*See Am. Math. Monthly, Feb. 1937, p. 102. Note that the exponent 2 for the numerator is missing there.—ED.

If the lines $A_ix+B_ix+C_i$, $(i=1,2,3)$, form a triangle, its area is given by

$$\frac{\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}^2}{2\cdot\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}\cdot\begin{vmatrix} A_2 & B_2 \\ A_3 & B_3 \end{vmatrix}\cdot\begin{vmatrix} A_3 & B_3 \\ A_1 & B_1 \end{vmatrix}}.$$

This theorem is not difficult for students of elementary determinant theory.

Another approach may be made upon the following well known theorem which is credited to *Archimedes*: If K represents the area formed by two tangents and their chord of contact and k the area between the chord and the parabola, then

$$\boxed{3k=2K} \tag{1}$$

Using the notation of *MacKay's* solution, we may write:

$$k_2=k_3+k_1+\Delta ABC,$$

$$K_2=K_3+K_1+\Delta ABC+\Delta DEF.$$

By (1), $\quad 3(k_3+k_1+\Delta ABC)=2(K_3+K_1+\Delta ABC+\Delta DEF),$

and thus $\quad \Delta ABC=2\cdot\Delta DEF.$

Also solved by *A. E. Gault, John Hallett, Fred Marer, W. T. Short, W. Irwin Thompson*, and *C. W. Trigg.*

No. 261. Proposed by *V. Thébault*, Le Mans, France.

Determine a perfect square corresponding to each of the following forms:

(1) *abc*578*abc*, (2) *abcabc*984, (3) 456*abcabc*.

Solution by *G. W. Wishard*, Norwood, Ohio.

(1) Noting that $578=2\cdot 17^2$, one may put immediately

$$(17\cdot 10^3+17)^2=289{,}578{,}289.$$

(2) Let $Y^2=abcabc984=984+abc\cdot 1000\cdot 1001$. Thus $Y^2\equiv 984$ mod 1001, from which follow $Y^2\equiv 4$ mod 7, $Y^2\equiv 5$ mod 11 and $Y^2\equiv 9$ mod 13, and hence $Y\equiv \pm 2$ mod 7, $Y\equiv \pm 4$ mod 11 and $Y\equiv \pm 3$ mod 13. Furthermore $Y^2\equiv 984$ mod 1000, from which follows $Y\equiv \pm 228$ mod 500. Evidently $10{,}000<Y<32{,}000$. Testing the values which satisfy these last two conditions against the requirements moduli 7, 11 and 13, one obtains the three solutions:

$$15272^2=233233984,\ 23228^2=539539984, \text{ and } 28772^2=827827984.$$

(3) Let $Z^2=456abcabc$. By the above method one finds $Z\equiv \pm 1$ mod 7, $Z\equiv \pm 4$ mod 11 and $Z\equiv \pm 1$ mod 13. Also $21354<Z<21380$. 21358 is the only number within the limits that yields the proper residues moduli 7, 11 and 13. $21358^2=456164164$.

The *Proposer's* solution for (1) was $26918^2=724578724$. That this and the solution given above are the only values was shown in the solution by *Fred Marer*. From the relation

$$X^2=578000+abc(10^6+1)=abc\cdot 101\cdot 9901$$

we have $X^2\equiv 78$ mod 101 and $X^2\equiv 3742$ mod 9901. The second of these congruences is tedious to solve, but when one solution is known the others are easy to find: since 9901 is prime, there exists a number g such that $X\equiv \pm g$ mod 9901 gives all the values. From the value 17017 for X given above we obtain $g=2785$, so that $X=9901\,u\pm 2785$. With $10000<X<31624$, u must be 1, 2 or 3, and the possible values of X are 12686, 17017, 22587 and 26918. Of these only 17017 and 26918 satisfy the congruence mod 101.

Also solved by *Johannes Mahrenholz* and *C. W. Trigg.*

No. 262. Proposed by *V. Thébault*, Le Mans, France.

Form a perfect square, the product of whose six digits is 190512.

Solution by *C. W. Trigg*, Los Angeles City College.

In the decimal scale $190512=2^4\cdot 3^5\cdot 7^2=6\cdot 7\cdot 7\cdot 8\cdot 9\cdot 9$, which is the only sextet of digits which may be formed from all the exhibited prime factors. Thus the desired square, N^2, ends with 76, 96, 89 or 69, and hence N is of the form $50k\pm a$, where $a=24$, 14, 17, or 13. Further, since the sum of the digits of N^2 is 46, $N^2\equiv 46\equiv 1$ mod 9, and thus $N\equiv \pm 1$ mod 9. With these conditions the eligible values of N are re-

duced to eight, of which only $883^2 = 779689$ and $937^2 = 877969$ exhibit the proper digits in N^2.

Also solved by *Albert Farnell, Lucille Meyer, G. W. Wishard, Fred Marer* and the *Proposer.*

No. 263. Proposed by *C. N. Mills,* Illinois State Normal University.

Given the sequence of terms

$$(n^2-2n+2),\ (n^2-4n+6),\ (n^2-6n+12),$$
$$(n^2-8n+20),\ (n^2-10n+30),\cdots,$$

determine values of n such that each of the first $(n-1)$ terms of the sequence is prime.

Solution by *C. W. Trigg,* Los Angeles City College.

The jth term of the sequence is $n^2-2jn+j(j+1)$. Thus the $(n-1-k)$th term reduces to k^2+k+n, and the first $n-1$ terms of the sequence are given by the values of k^2+k+n for $k=0,1,2,\cdots, n-2$. The first six values of n for which these terms are all primes and the corresponding terms are:

$n=2$: 2. $n=3$: 5, 3. $n=5$: 17, 11, 7, 5.

$n=11$: 101, 83, 67, 53, 41, 31, 23, 17, 13, 11.

$n=17$: 257, 227, 199, 173, 149, 127, 107, 89, 73, 59, 47, 37, 29, 23, 19, 17.

$n=41$: 1601, 1523, 1447, 1373, 1301, 1231, 1163, 1097, 1033, 971, 911, 853, 797, 743, 691, 641, 593, 547, 503, 461, 421, 383, 347, 313, 281, 251, 223, 197, 173, 151, 131, 113, 97, 83, 71, 61, 53, 47, 43, 41.

There seems little likelihood of any further values. It is stated in L. E. Dickson's *History of the Theory of Numbers*, Vol. I, p. 420: "H. LeLasseur verified that, for a prime A between 41 and 54,000, x^2+x+A does not represent primes exclusively for $x=0,1,\cdots, A-2$. $\cdots$ Escott examined values of A much exceeding 54000 without finding a suitable $A>41$."

Also solved by *Johannes Mahrenholz, Fred Marer* and *G. W. Wishard.*

No. 264. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries.

Let $T_n = \sqrt[3]{5+2T_{n-1}}$, $T_0 = \sqrt[3]{5}$,

$S_n = \sqrt{(5+2S_{n-1})/S_{n-1}}$, $S_0 = \sqrt{2}$.

Prove that: $\lim\limits_{n\to\infty} T_n = \lim\limits_{n\to\infty} S_n$

Solution by *W. E. Byrne*, Virginia Military Institute.

The solution may be based upon the following theorem: *If the equation* $x = f(x)$ *has only one root in the interval* (a,b) *and if* $|f'(x)| < k < 1$ *in* (a,b), *then the sequence of numbers* $x_0, x_1, \cdots, x_n, \cdots$ defined by

$$a < x_0 < b, \quad x_n = f(x_{n-1}), \qquad n = 1,2,\cdots$$

has a limit $\bar{x}$ *such that* $\bar{x} = f(\bar{x})$. (*Cours de Mathématiques Spéciales*, Commissaire et Cagnac, Tome II, page 230).

We may take $a=1$, $b=3$ for both sequences. The condition to be imposed on the derivatives is satisfied. Let $S = \lim S_n$, and $T = \lim T_n$. Then S is a root of the equation

$$S = \sqrt{(5+2S)/S}, \qquad S^3 = 5+2S$$

and T is a root of

$$T = \sqrt[3]{5+2T}, \qquad T^3 = 5+2T.$$

Hence $S = T$. The sequence T_n is monotonic increasing, while the successive S_n are alternately smaller and larger than S.

Also solved by *Fred Marer* and the *Proposer*.

PROPOSALS

No. 291. Proposed by *W. E. Byrne*, Virginia Military Institute.

Show that among all the integral curves of

$$xp^2 + yp + 1 = 0, \quad p = dy/dx,$$

(1) there is one parabola;

(2) the p-discriminant locus (with the exception of the origin) is the locus of cusps of the general integral curves;

(3) the general integral curves admit the y-axis as an asymptote;

(4) the y-axis and the parabola of (1) form the limiting curves of general integral curves (whose cusps are not at the origin).

No. 292. Proposed by *Daniel Arany*, Budapest, Hungary.

Establish the following identity:

$$\sin^2 x = \sin^2 y + \sin^2(x+y) - 2 \sin y \sin(x+y) \cos x.$$

No. 293. Proposed by *Howard D. Grossman*, New York City.

Nim is a game described in books on mathematical recreations. Small similar objects are divided into any number of groups with any number in each group. Two players play alternately, each removing from any one group any positive number of objects. He who removes the last of all the objects wins the game.

A solution is known to be as follows: Express at each move the number of objects in each group in the scale of 2 and arrange these numbers one under another; then play so as to leave the sum of the digits in each column an even number. If a player once meets this condition, his opponent must destroy it and he can restore it. Prove this solution.

Prove also this generalization: If the game is played so that any positive number of objects may be removed from any number of groups $\leqq k$, then the solution is, expressing and arranging the numbers as before, to leave the sum of the digits in each column $\equiv 0 \mod (k+1)$. If a player once meets this condition, his opponent must destroy it and he can restore it.

No. 294. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a square whose sides, prolonged if necessary, will each pass throuth one of four arbitrary points in a plane. How many such squares exist?

No. 295. Proposed by *V. Thébault*, Le Mans, France.

In the system of numeration with base 31, form all perfect squares of four digits having the form *aabb*. What notable properties appear in certain of these?

No. 296. Proposed by *W. V. Parker*, Louisiana State University.

Prove that the ellipse of least area circumscribing any right triangle has eccentricity not less than $\sqrt{(2/3)}$.

No. 297. Proposed by *H. T. R. Aude,* Colgate University.

Each of the two equations

$$30x+23y=c,$$

$$30x+23y=c+1$$

has only one solution in positive integers. Show that the greatest value of c is more than three times the least value of c.

No. 298. Proposed by *Howard D. Grossman,* New York City.

Let a,b,c be the sides of any triangle, G its centroid, I the incenter, and P the centroid of its perimeter. If parallels to the sides be drawn through G, any side b is divided *internally* in the ratio $1:1:1$; if through I, in the ratio $a:b:c$; if through P, in the ratio $(b+c):(c+a):(a+b)$.

No. 299. Proposed by *R. E. Gaines,* University of Richmond.

A slender rod of length $2l$ rests on a square table the length of whose side is $2a$ $(l<a)$. Find the probability (1) that both ends of the rod are on the table and (2) that both ends of the rod extend over the edge of the table.

Correction: The editors regret the error in the statement of No. 279, February, 1939, Vol. XIII, 5, p. 251. For "$27pqr+s^3=0$" read "$27pqr+s^3\neq 0$". (Note that the condition $27pqr+s^3=0$ makes the cubic the product of three linear factors, so that the general solution becomes trivial).

Bibliography and Reviews

Edited by
P. K. SMITH and H. A. SIMMONS

Mathematical Snapshots. By H. Steinhaus. G. E. Stechert and Company, New York, 1938. 135 pages. $2.50. (Translated from the Polish).

This is one of the most delightful books on mathematics to appear in many years. In my opinion it should be made available to all students of grade school, high school, and college, for pure enjoyment and for the sake of the stimulation it will undoubtedly afford. As the author says, "the book appeals to the scientist in the child and to the child in the scientist".

The underlying theme, and the only apparent one, is the presentation of theorems and bits of mathematical information, objectively and pictorially. In this the author and his small corps of assistants have indeed been successful. Although there is no system indicated in the selection of topics, quite frequently one discussion leads naturally to the next following.

Among the many features will be found chess problems, the mathematics of music, nomograms, the path of a billiard ball, continued fractions, tessellations, constructions of the compasses, perspectivity, roulettes, Minkowski's theorem, closed figures of constant breadth, conic sections, Sierpinski's curve that fills a square, regular polyhedra and space filling, soap bubbles, world maps, the conoids, topology, Möbius bands, the four-color map problem, Pascal's number triangle, and the Gauss curve.

The ingenuity displayed in the drawings and photographs, which appear on every page, is remarkable. The variety of illustrations of the Gauss curve in nature, the composite photograph of the catenary and soap film, those of the Möbius bands, the knotted ropes of topology, the cusped shadow of a twisted wire, etc., all contribute to the unusual charm of the book.

The anaglyphs, to be studied with the red and green spectacles found pocketed in the back,* are disappointing. They are quite

*In the same pocket will be found a set of cards which give animated pictures of the path of a projectile, the famous brachistochrone problem, the path of the earth about the sun, and the line motion generated by the internal rolling of two wheels. There is also a colored dodecahedron which, believe-it-or-not, collapses neatly into the pocket.

indistinct and do not give the intended effect. Another minor criticism should be directed to the midpoint construction of p. 38. Eight circles are used where only seven are necessary.

The publishers are to be highly commended on this excellent construction job. The book, printed in several colors, is evidently the result of great care and considerable hand work.

University of Maryland. ROBERT C. YATES.

Analytic Geometry. By Roscoe Woods. The Macmillan Company, New York, 1939. xiii+294 pages. $2.25.

The first twelve chapters (223 pages) of the text under review are devoted to plane analytic geometry; the remaining three chapters (53 pages) to solid analytic geometry. The first chapter introduces the fundamental concepts and formulas concerning slopes, length of line segments, parallelism and perpendicularity of lines, etc. The next five chapters take up in order the straight line, circle, parabola, ellipse, and hyperbola with a very excellent balance of emphasis on these topics. Chapter VII deals with transformations of coordinates; Chapter VIII, with the discussion of curves from their equations. Polar coordinates are first introduced in Chapter IX and are very nicely discussed. Parametric equations are discussed in Chapter X. Chapters XII and XIII are devoted to other geometric properties of conics.

The discussion of solid analytic geometry is confined to the usual topics found in most elementary texts, including the usual concepts of lengths, direction cosines, coordinate systems, lines, planes, and the quadric surfaces.

As stated in the preface, the text is adapted to a short or a long course, the adaptation being accomplished by placing the more fundamental prerequisites for the calculus in the first ten chapters (182 pages); these chapters form a minimum course in the subject.

The author has deferred polar coordinates to Chapter IX. He states in the preface that this was done intentionally; that it is his belief that the student gains by using polar coordinates without reference to any other (rectangular) system. I am sure that many authors and teachers do not agree with this point of view but prefer to carry the two systems along together.

There are a few innovations. For example, on page 39 direction cosines of a line, considered as a locus in a plane, are introduced.

By defining a tangent to a conic as a line having two coincident points of intersection with the conic, the author develops the equations

of the tangents to the conics, first in terms of the slopes and then in terms of the points of contact. In that manner, he delays the limit concept until the study of the calculus.

A great deal of material usually found as textual material appears in the very excellent lists of problems. These lists of problems are well graded; there are sufficient easy problems for the poorer student and plenty of others to offer intellectual satisfaction to the superior student.

There are a few errors in the text. At the top of page 31, problem 7, the conditions that two lines be parallel are necessary but not sufficient. Similarly, the same error is made on page 242 concerning the conditions that two planes be parallel. In the answer to problem 29, page 106, the coordinates axes are not specified.

A more serious error occurs in discussing a locus from its equation in the loose use of the term *complex*. This error appears in particular in the note near the bottom of page 53; in the discussions of examples 1, 3, 4, 5, pages 127, 128, 129, 130; and in examples 1, 2, pages 130, 131. The author uses the term *complex* as synonymous with non-real (imaginary). It is just as easy to use the customary terminology and this does not get the teacher in trouble who insists on the usual distinctions.

Altogether the book appears to be teachable; it will no doubt be a stimulating book to the good student and a comprehensible book to the poor pupil. The text is nicely bound; the print is easy to read, and the important facts are shown by italics or clarendon type.

Michigan State College. V. G. GROVE.

Introduction to the Theory of Groups of Finite Order. By Robert D. Carmichæl. Ginn and Company, Boston, 1937. xiv+175 pages. $5.00.

This book admirably achieves the first stated purpose of the author, namely, to give "an exposition which first of all prepares him for the development of the theory and then rapidly introduces him to a few fundamental theorems by which the construction of a large part of the theory may be effected". Emphasis is rightly placed on definitions and proofs and interrelations of the important abstract ideas of the theory. However, the abstract concepts are introduced and illustrated by examples. There are frequent lists of numerous problems which should be valuable to readers and teachers meeting the subject for the first time in independent study. A feature of the book is the careful summarizing in theorems of all important results. The display of these theorems on the page, and the large type used in both theorems and discussion are noteworthy.

The outstanding merit of the book, in the reviewer's opinion, is the presentation, with great clarity and in less than one hundred pages, of the five fundamental theorems on finite groups and the fundamental results on isomorphism of finite groups. The stated definition of isomorphism involves the undefined term "corresponds". The idea of isomorphism is that of function, and an adequate notation would clarify this important idea and simplify the proofs. Such a notation leads also to an easier understanding of recent research on isomorphism and homomorphism, not only for groups but for other abstractly defined systems.

The presentation is unusually fine of the idea of holomorph of a group, and of the very fundamental facts on factor groups and the Jordan-Hölder theorem. There are long chapters on Abelian groups, on prime-power groups, and on groups of linear transformations, and introductory chapters on Galois fields, finite geometries, and certain algebras of doubly transitive groups.

Northwestern University. L. W. GRIFFITHS.

Announcement!

Beginning with Volume XIV, October 1, 1939, the subscription price of NATIONAL MATHEMATICS MAGAZINE will be increased to $2.00 per volume, or per year.

The following special concession will be made: If at any time *previous to October 1, 1939,* a new subscription is paid into the Baton Rouge office to cover a period of *one* year, or *two* years, or *three* years, it will be sufficient to remit at the old rate of $1.50 per year.

Again, any subscriber whose subscription is paid up to October 1, 1939, and who desires a *year,* or *two* years, or *three* years subscription additional, may secure this subscription at the old rate of $1.50 per year, PROVIDED money for such subscription is paid into our Baton Rouge office BEFORE October, 1939.

For subscriptions received after October 1, 1939, to cover periods beginning after this date, a $2.00 subscription rate will be charged.

Editor and Manager
NATIONAL MATHEMATICS MAGAZINE